Entanglement in continuous-variable systems: recent advances and current perspectives

2007 J. Phys. A: Math. Theor. 40 7821

(http://iopscience.iop.org/1751-8121/40/28/S01)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 7821–7880 doi:10.1088/1751-8113/40/28/S01

Entanglement in continuous-variable systems: recent advances and current perspectives

Gerardo Adesso[1,2,3] **and Fabrizio Illuminati**[2,3,4]

[1] Dipartimento di Fisica, Università degli Studi di Roma 'La Sapienza', Piazzale Aldo Moro 5, I-00185 Roma, Italy
[2] Dipartimento di Matematica e Informatica, Università degli Studi di Salerno, Via Ponte don Melillo, I-84084 Fisciano (SA), Italy
[3] CNR-INFM Coherentia, Napoli, Italy and INFN Sezione di Napoli, Gruppo collegato di Salerno, Baronissi (SA), Italy
[4] ISI Foundation for Scientific Interchange, Viale Settimio Severo 65, I-10133 Torino, Italy

Received 23 January 2007
Published 27 June 2007
Online at stacks.iop.org/JPhysA/40/7821

Abstract

We review the theory of continuous-variable entanglement with special emphasis on foundational aspects, conceptual structures and mathematical methods. Much attention is devoted to the discussion of separability criteria and entanglement properties of Gaussian states, for their great practical relevance in applications to quantum optics and quantum information, as well as for the very clean framework that they allow for the study of the structure of nonlocal correlations. We give a self-contained introduction to phase-space and symplectic methods in the study of Gaussian states of infinite-dimensional bosonic systems. We review the most important results on the separability and distillability of Gaussian states and discuss the main properties of bipartite entanglement. These include the extremal entanglement, minimal and maximal, of two-mode mixed Gaussian states, the ordering of two-mode Gaussian states according to different measures of entanglement, the unitary (reversible) localization and the scaling of bipartite entanglement in multimode Gaussian states. We then discuss recent advances in the understanding of entanglement sharing in multimode Gaussian states, including the proof of the monogamy inequality of distributed entanglement for all Gaussian states. Multipartite entanglement of Gaussian states is reviewed by discussing its qualification by different classes of separability, and the main consequences of the monogamy inequality, such as the quantification of genuine tripartite entanglement in three-mode Gaussian states, the promiscuous nature of entanglement sharing in symmetric Gaussian states and the possible coexistence of unlimited bipartite and multipartite entanglement. We finally review recent advances and discuss possible perspectives on the qualification and quantification of entanglement in non-Gaussian states, a field of research that is to a large extent yet to be explored.

PACS numbers: 03.67.Mn, 03.65.Ud

(Some figures in this article are in colour only in the electronic version)

Contents

1. Prologue

About 80 years after their inception, quantum mechanics and quantum theory are still an endless source of new and precious knowledge on the physical world and at the same time keep evolving in their mathematical structures, conceptual foundations, and intellectual and cultural implications. This is one of the reasons why quantum physics is still so specially fascinating to all those that approach it for the first time and never ceases to be so for those that are professionally involved with it. In particular, since the early 1990s of the last century and in the last 10–15 years, a quiet revolution has taken place in the quantum arena. This revolution has progressively indicated and clarified that aspects once thought to be problematic, such as quantum nonseparability and 'spooky' actions at a distance, are actually not only problems but rather some of the key ingredients that are allowing a deeper understanding of quantum mechanics, its applications to new and exciting fields of research (such as quantum information and quantum computation) and tremendous progress in the development of its mathematical and conceptual foundations. Among the key elements of the current re-foundation of quantum mechanics, entanglement certainly plays a very important role because it is a concept that can be mathematically qualified and quantified in a way that allows it to provide new and general characterizations of quantum properties, operations and states.

In the context of this special issue on *quantum information*, we will review the main aspects of entanglement in continuous-variable systems, as they are currently understood. With the aim to be as self-contained as possible, we start with a tutorial summary of the structural properties of continuous-variable systems—systems associated with infinite-dimensional Hilbert spaces whose possible applications in quantum information and communication tasks are gaining increasing interest and attention—focusing on the specially relevant family of Gaussian states. The reason why this review is mainly focused on the discussion of separability criteria and entanglement properties of Gaussian states is due to their great practical relevance in applications to quantum optics and quantum information, to the very clean framework that they allow for the study of the structure of nonlocal correlations and to the obvious consequence that in the last years most studies and results on continuous-variable entanglement have been obtained for Gaussian states.

In sections 2 and 3, we give a self-contained introduction to phase-space and symplectic methods in the study of Gaussian states of infinite-dimensional bosonic systems, we discuss the covariance matrix formalism and we provide a classification of pure and mixed Gaussian states according to the various forms that the associated covariance matrices can take. In section 4, we introduce and describe the separability problem, the inseparability criteria and the machinery of bipartite entanglement evaluation and distillation in Gaussian states. In section 5, we review some important results specific on two-mode Gaussian states, including the existence of extremally entangled states, minimal and maximal, at a given degree of mixedness, and the different orderings induced on the set of two-mode Gaussian states by different measures of entanglement such as the Gaussian entanglement of formation and the logarithmic negativity. In section 6, we describe the unitary (and therefore reversible) localization of bipartite multimode entanglement to bipartite two-mode entanglement in fully symmetric and bisymmetric multimode Gaussian states, and the scaling of bipartite entanglement with the number of modes in general multimode Gaussian states. In section 7, we then discuss recent crucial advances in the understanding of entanglement sharing in multimode Gaussian states, including the proof of the monogamy inequality of distributed entanglement for all Gaussian states. Multipartite entanglement of Gaussian states is reviewed in section 8 by discussing its qualification according to different classes of separability, and the main consequences of the monogamy inequality, such as the quantification of genuine tripartite entanglement in three-mode Gaussian states via the residual Gaussian tangle, the promiscuous nature of entanglement sharing in Gaussian states with symmetry constraints and the possible coexistence of unlimited bipartite and multipartite entanglement. The last two properties (promiscuity and coexistence of unbound bipartite and multipartite entanglement) are elucidated at length by discussing three- and four-mode Gaussian states endowed with full and partial symmetry constraints under the exchange of modes. Section 9 concludes this review with a brief discussion about recent advances in the qualification and quantification of entanglement in non-Gaussian states, a field of research that is to a large extent yet to be fully explored, a summary on various applications of Gaussian entanglement in quantum information and computation, and on overview on open problems and current research directions.

2. Introduction to continuous-variable systems

A continuous-variable (CV) system [1–3] of N canonical bosonic modes is described by a Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$ resulting from the tensor product structure of infinite-dimensional Fock spaces $\mathcal{H}_k$'s, each of them associated with a single mode. For instance, one can think of the noninteracting quantized electromagnetic field, whose Hamiltonian describes a system of an arbitrary number N of harmonic oscillators of different frequencies, the *modes* of the field,

$$\hat{H} = \sum_{k=1}^{N} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right). \tag{1}$$

Here, $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and creation operators of a photon in mode k (with frequency ω_k), which satisfy the bosonic commutation relation

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta_{kk'}, \qquad [\hat{a}_k; \hat{a}_{k'}] = \left[m\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0. \tag{2}$$

From now on we will assume for convenience natural units with $\hbar = 2$. The corresponding quadrature phase operators (position and momentum) for each mode are defined as

$$\hat{q}_k = \left(\hat{a}_k + \hat{a}_k^\dagger\right), \tag{3}$$

$$\hat{p}_k = \left(\hat{a}_k - \hat{a}_k^\dagger\right)/\mathrm{i}. \tag{4}$$

We can group together the canonical operators in the vector

$$\hat{R} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\mathsf{T}, \tag{5}$$

which enables us to write in compact form the bosonic commutation relations between the quadrature phase operators,

$$[\hat{R}_k, \hat{R}_l] = 2\mathrm{i}\Omega_{kl}, \tag{6}$$

where Ω is the symplectic form

$$\Omega = \bigoplus_{k=1}^{N} \omega, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7}$$

The space $\mathcal{H}_k$ is spanned by the Fock basis $\{|n\rangle_k\}$ of eigenstates of the number operator $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$, representing the Hamiltonian of the noninteracting mode via equation (1). The Hamiltonian of each mode is bounded from below, thus ensuring the stability of the system. For each mode k there exists a different vacuum state $|0\rangle_k \in \mathcal{H}_k$ such that $\hat{a}_k|0\rangle_k = 0$. The vacuum state of the global Hilbert space will be denoted by $|0\rangle = \bigotimes_k |0\rangle_k$. In the single-mode

Hilbert space $\mathcal{H}_k$, the eigenstates of $\hat{a}_k$ constitute the important set of *coherent states* [4], which is overcomplete in $\mathcal{H}_k$. Coherent states result from applying the single-mode Weyl displacement operator $\hat{D}_k$ to the vacuum $|0\rangle_k$, $|\alpha\rangle_k = \hat{D}_k(\alpha)|0\rangle_k$, where

$$\hat{D}_k(\alpha) = \mathrm{e}^{\alpha\hat{a}_k^\dagger - \alpha^*\hat{a}_k}, \tag{8}$$

and the coherent amplitude $\alpha \in \mathbb{C}$ satisfies $\hat{a}_k|\alpha\rangle_k = \alpha|\alpha\rangle_k$. In terms of the Fock basis of mode k, a coherent state reads

$$|\alpha\rangle_k = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_k. \tag{9}$$

Tensor products of coherent states for N different modes are obtained by applying the N-mode Weyl operators $\hat{D}_\xi$ to the global vacuum $|0\rangle$. For future convenience, we define the operators D_ξ in terms of the canonical operators $\hat{R}$,

$$\hat{D}_\xi = \mathrm{e}^{i\hat{R}^\mathsf{T}\Omega\xi}, \qquad \text{with} \quad \xi \in \mathbb{R}^{2N}. \tag{10}$$

One then has $|\xi\rangle = \hat{D}_\xi|0\rangle$.

2.1. Quantum phase-space picture

The states of a CV system are the set of positive trace-class operators $\{\varrho\}$ on the Hilbert space $\mathcal{H} = \bigotimes_{k=1}^N \mathcal{H}_k$. However, the complete description of any quantum state ϱ of such an infinite-dimensional system can be provided by one of its s-ordered *characteristic functions* [5]

$$\chi_s(\xi) = \mathrm{Tr}[\varrho\hat{D}_\xi]\,\mathrm{e}^{s\|\xi\|^2/2}, \tag{11}$$

with $\xi \in \mathbb{R}^{2N}$, $\|\cdot\|$ standing for the Euclidean norm of $\mathbb{R}^{2N}$. The vector ξ belongs to the real $2N$-dimensional space $\Gamma = (\mathbb{R}^{2N}, \Omega)$, which is called *phase space*, in analogy with classical Hamiltonian dynamics. One can see from the definition of the characteristic functions that in the phase space picture the tensor product structure is replaced by a direct sum structure, so that the N-mode phase space $\Gamma = \bigoplus_k \Gamma_k$, where $\Gamma_k = (\mathbb{R}^2, \omega)$ is the local phase space associated with mode k.

The family of characteristic functions is in turn related, via complex Fourier transform, to the *quasi-probability distributions* W_s, which constitute another set of complete descriptions of the quantum states

$$W_s(\xi) = \frac{1}{\pi^2}\int_{\mathbb{R}^{2N}} \kappa\,\chi_s(\kappa)\,\mathrm{e}^{\mathrm{i}\kappa^\mathsf{T}\Omega\xi}\,\mathrm{d}^{2N}. \tag{12}$$

There exist states for which the function W_s is not a regular probability distribution for any s, because it can be singular or assume negative values. Note that the value $s = -1$ corresponds to the Husimi 'Q-function' [6] $W_{-1}(\xi) = \langle\xi|\varrho|\xi\rangle/\pi$ and thus always yields a regular probability distribution. The case $s = 0$ corresponds to the so-called Wigner function [7], which will be denoted simply by W. Likewise, for the sake of simplicity, χ will stand for the symmetrically ordered characteristic function χ_0. Finally, the case $s = 1$ yields the singular P-representation, which was introduced, independently, by Glauber [8] and Sudarshan [9].

The quasi-probability distributions of integer order W_{-1}, W_0 and W_1 are respectively associated the antinormally ordered, symmetrically ordered and normally ordered expressions of operators. More precisely, if the operator $\hat{O}$ can be expressed as $\hat{O} = f\left(\hat{a}_k, \hat{a}_k^\dagger\right)$ for $k = 1, \ldots, N$, where f is a, say, symmetrically ordered function of the field operators, then one has [10, 11]

$$\mathrm{Tr}[\varrho\hat{O}] = \int_{\mathbb{R}^{2N}} W_0(\kappa)\bar{f}(\kappa)\,\mathrm{d}^{2N}\kappa,$$

Table 1. Schematic comparison between Hilbert-space and phase-space pictures for N-mode continuous-variable systems.

	Hilbert space $\mathcal{H}$	Phase space Γ
Dimension	∞	$2N$
Structure	$\otimes$	$\oplus$
Description	ϱ	χ_s, W_s

where $\bar{f}(\kappa) = f(\kappa_k + \mathrm{i}\kappa_{k+1}, \kappa_k - \mathrm{i}\kappa_{k+1})$ and f takes the same form as the operatorial function previously introduced. The same relationship holds between W_{-1} and the antinormally ordered expressions of the operators and between W_1 and the normal ordering. We also recall that the normally ordered function of a given operator is provided by its Wigner representation. This entails the following equalities for the trace

$$1 = \mathrm{Tr}\,\varrho = \int_{\mathbb{R}^{2N}} W(\kappa)\,\mathrm{d}^{2N}\kappa = \chi(0), \tag{13}$$

and for the purity [12]

$$\mu = \mathrm{Tr}\,\varrho^2 = \int_{\mathbb{R}^{2N}} W^2(\kappa)\,\mathrm{d}^{2N}\kappa = \int_{\mathbb{R}^{2N}} |\chi(\xi)|^2\,\mathrm{d}^{2N}\xi \tag{14}$$

of a state ϱ. These expressions will be useful in the following.

The (symmetric) Wigner function can be written as follows in terms of the (unnormalized) eigenvectors $|x\rangle$ of the quadrature operators $\{\hat{q}_j\}$ (for which $\hat{q}_j|x\rangle = q_j|x\rangle, x \in \mathbb{R}^N$, for $j = 1, \ldots, N$) [13]

$$W(x, p) = \frac{1}{\pi^N}\int_{\mathbb{R}^N} \langle x - x'|\varrho|x + x'\rangle\,\mathrm{e}^{\mathrm{i}x'\cdot p}\,\mathrm{d}^N x', \qquad x, p \in \mathbb{R}^N. \tag{15}$$

From an operational point of view, the Wigner function admits a clear interpretation in terms of homodyne measurements [14]: the marginal integral of the Wigner function over the variables $p_1, \ldots, p_N, x_1, \ldots, x_{N-1}$

$$\int_{\mathbb{R}^{2N-1}} W(x, p)\,\mathrm{d}^N p\,\mathrm{d}x_1 \cdots \mathrm{d}x_{N-1}$$

gives the probability of the results of homodyne detections on the remaining quadrature x_N [15].

Table 1 summarizes the mapping of properties and tools between Hilbert and phase spaces. In the next section, more properties and tools of phase space analysis will be introduced that are specially useful in the study of Gaussian states.

3. Mathematical description of Gaussian states

The set of *Gaussian states* is, by definition, the set of states with Gaussian characteristic functions and quasi-probability distributions on the multimode quantum phase space. Gaussian states include, among others, coherent, squeezed and thermal states. Therefore, they are of central importance in quantum optics and in quantum information and quantum communication with CV systems [16]. Their entanglement properties will thus be one of the main subjects of this review.

3.1. Covariance matrix formalism

From the definition it follows that a Gaussian state ϱ is completely characterized by the first and second statistical moments of the quadrature field operators, which will be denoted, respectively, by the vector of first moments $\bar{R} = (\langle\hat{R}_1\rangle, \langle\hat{R}_1\rangle, \ldots, \langle\hat{R}_N\rangle, \langle\hat{R}_n\rangle)$ and by the covariance matrix (CM) $\boldsymbol{\sigma}$ of elements

$$\boldsymbol{\sigma}_{ij} = \tfrac{1}{2}\langle\hat{R}_i\hat{R}_j + \hat{R}_j\hat{R}_i\rangle - \langle\hat{R}_i\rangle\langle\hat{R}_j\rangle. \tag{16}$$

First moments can be arbitrarily adjusted by local unitary operations, namely displacements in phase space, i.e. applications of the single-mode Weyl operator, equation (8), to locally re-centre the reduced Gaussian corresponding to each single mode (recall that the reduced state obtained from a Gaussian state by partial tracing over a subset of modes is still Gaussian). Such operations leave any informationally relevant property, such as entropy and entanglement, invariant. Therefore, from now on (unless otherwise stated) we will adjust first moments to 0 without any loss of generality for the scopes of our analysis.

With this position, the Wigner function of a Gaussian state can be written as follows in terms of phase-space quadrature variables:

$$W(X) = \frac{\mathrm{e}^{-\frac{1}{2}X\boldsymbol{\sigma}^{-1}X^{\mathsf{T}}}}{\pi\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}}, \tag{17}$$

where R stands for the real phase-space vector $(q_1, p_1, \ldots, q_n, p_n) \in \Gamma$. Therefore, despite the infinite dimension of the associated Hilbert space, the complete description of an arbitrary Gaussian state (up to local unitary operations) is given by the $2N \times 2N$ CM $\boldsymbol{\sigma}$. In the following, $\boldsymbol{\sigma}$ will be assumed indifferently to denote the matrix of second moments of a Gaussian state or the Gaussian state itself. In the language of statistical mechanics, the elements of the CM are the two-point truncated correlation functions between the $2N$ canonical continuous variables. We also note that the entries of the CM can be expressed as energies by multiplying them by the level spacing $\hbar\omega_k$, where ω_k is the frequency of each mode k. In this way, $\mathrm{Tr}\,\boldsymbol{\sigma}$ is related to the mean energy of the state, i.e. the average of the noninteracting Hamiltonian equation (1).

As the real $\boldsymbol{\sigma}$ contains the complete locally invariant information on a Gaussian state, we can expect some constraints to exist to be obeyed by any *bona fide* CM, reflecting in particular the requirements of positive semidefiniteness of the associated density matrix ϱ. Indeed, such condition together with the canonical commutation relations implies

$$\boldsymbol{\sigma} + \mathrm{i}\Omega \geqslant 0. \tag{18}$$

In equation (18) is the necessary and sufficient constraint the matrix $\boldsymbol{\sigma}$ has to fulfil to be a CM corresponding to a physical Gaussian state [17, 18]. More in general, the previous condition is necessary for the CM of *any*, generally non-Gaussian, CV state (characterized in principle by the moments of any order). We note that such a constraint implies $\boldsymbol{\sigma} \geqslant 0$. In equation (18) is the expression of the uncertainty principle on the canonical operators in its strong, Robertson–Schrödinger form [19–21].

For future convenience, let us define and write the CM $\boldsymbol{\sigma}_{1\ldots N}$ of an N-mode Gaussian state in terms of two-by-two submatrices as

$$\boldsymbol{\sigma}_{1\ldots N} = \begin{pmatrix} \boldsymbol{\sigma}_1 & \varepsilon_{1,2} & \cdots & \varepsilon_{1,N} \\ \varepsilon_{1,2}^{\mathsf{T}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon_{N-1,N} \\ \varepsilon_{1,N}^{\mathsf{T}} & \cdots & \varepsilon_{N-1,N}^{\mathsf{T}} & \boldsymbol{\sigma}_N \end{pmatrix}. \tag{19}$$

Each diagonal block $\boldsymbol{\sigma}_k$ is respectively the local CM corresponding to the reduced state of mode k, for all $k = 1, \ldots, N$. On the other hand, the off-diagonal matrices $\varepsilon_{i,j}$ encode the intermodal correlations (quantum and classical) between subsystems i and j. The matrices $\varepsilon_{i,j}$ all vanish for a product state.

In this preliminary overview, let us just mention an important instance of two-mode Gaussian state, the *two-mode squeezed state* $|\psi^{\rm sq}\rangle_{i,j} = \hat{U}_{i,j}(r)(|0\rangle_i \otimes |0\rangle_j)$ with squeezing factor $r \in \mathbb{R}$, where the (phase-free) two-mode squeezing operator is given by

$$\hat{U}_{i,j}(r) = \exp\left[-\frac{r}{2}\left(\hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{a}_i \hat{a}_j\right)\right]. \tag{20}$$

In the limit of infinite squeezing ($r \to \infty$), the state approaches the ideal Einstein–Podolsky–Rosen (EPR) state [22], simultaneous eigenstate of total momentum and relative position of the two subsystems, which thus share *infinite entanglement*. The EPR state is unnormalizable and unphysical. However, in principle, an EPR state can be approximated with an arbitrarily high degree of accuracy by two-mode squeezed states with sufficiently large squeezing. Therefore, two-mode squeezed states are of key importance as entangled resources for practical implementations of CV quantum information protocols [1]. They thus play a central role in the subsequent study of the entanglement properties of Gaussian states. A two-mode squeezed state with squeezing r (also known in quantum optics as a *twin-beam* state) is described by a CM

$$\boldsymbol{\sigma}^{\rm sq}_{i,j}(r) = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix}. \tag{21}$$

3.2. Symplectic operations

An important role in the theoretical and experimental manipulation of Gaussian states is played by unitary operations which preserve the Gaussian character of the states on which they act. These unitary Gaussian operations are all those generated by Hamiltonian terms at most quadratic in the field operators. As a consequence of the Stone–Von Neumann theorem, the so-called *metaplectic* representation entails that any such unitary operation at the Hilbert space level corresponds, in phase space, to a symplectic transformation, i.e. to a linear transformation S which preserves the symplectic form Ω:

$$S^{\mathsf{T}}\Omega S = \Omega. \tag{22}$$

Symplectic transformations on a $2N$-dimensional phase space form the (real) symplectic group $Sp_{(2N,\mathbb{R})}$. Such transformations act linearly on first moments and by congruence on covariance matrices, $\boldsymbol{\sigma} \mapsto S\boldsymbol{\sigma}S^{\mathsf{T}}$. Equation (22) implies $\mathrm{Det}\, S = 1$, $\forall S \in Sp_{(2N,\mathbb{R})}$. Ideal beam splitters, phase shifters and squeezers are all described by some kind of symplectic transformation (see, e.g., [15]). For instance, the two-mode squeezing operator equation (20) corresponds to the symplectic transformation

$$S_{i,j}(r) = \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}, \tag{23}$$

where the matrix is understood to act on the pair of modes i and j. In this way, the two-mode squeezed state, equation (21), can be obtained as $\boldsymbol{\sigma}^{\rm sq}_{i,j}(r) = S_{i,j}(r)\mathbb{1}S^{\mathsf{T}}_{i,j}(r)$ exploiting the fact that the CM of the two-mode vacuum state is the 4×4 identity matrix.

Another common unitary operation is the ideal (phase-free) *beam splitter*, whose action $\hat{B}_{i,j}$ on a pair of modes i and j is defined as

$$\hat{B}_{i,j}(\theta) : \begin{cases} \hat{a}_i \to \hat{a}_i \cos\theta + \hat{a}_j \sin\theta \\ \hat{a}_j \to \hat{a}_i \sin\theta - \hat{a}_j \cos\theta, \end{cases} \tag{24}$$

with $\hat{a}_l$ being the annihilation operator of mode k. A beam splitter with transmittivity τ corresponds to a rotation of $\theta = \arccos\sqrt{\tau}$ in phase space ($\theta = \pi/4$ corresponds to a balanced 50:50 beam splitter, $\tau = 1/2$), described by a symplectic transformation

$$B_{i,j}(\tau) = \begin{pmatrix} \sqrt{\tau} & 0 & \sqrt{1-\tau} & 0 \\ 0 & \sqrt{\tau} & 0 & \sqrt{1-\tau} \\ \sqrt{1-\tau} & 0 & -\sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} & 0 & -\sqrt{\tau} \end{pmatrix}. \tag{25}$$

Single-mode symplectic operations are easily introduced as linear combinations of planar (orthogonal) rotations and of single-mode squeezings of the form

$$S_j(r) = \mathrm{diag}(\mathrm{e}^r, \mathrm{e}^{-r}), \tag{26}$$

acting on mode j, for $r > 0$.

In general, symplectic transformations in phase space are generated by the exponentiation of matrices written as $J\Omega$, where J is antisymmetric [23]. Such generators can be symmetric or antisymmetric. The operations $B_{ij}(\tau)$, equation (25), generated by antisymmetric operators are orthogonal and, acting by congruence on the CM $\boldsymbol{\sigma}$, preserve the value of $\mathrm{Tr}\,\boldsymbol{\sigma}$. Since $\mathrm{Tr}\,\boldsymbol{\sigma}$ gives the contribution of the second moments to the average of the Hamiltonian $\bigoplus_k \hat{a}_k^\dagger \hat{a}_k$, these transformations are said to be *passive* (they belong to the compact subgroup of $Sp_{(2N,\mathbb{R})}$). Instead, operations $S_{i,j}(r)$, equation (23), generated by symmetric operators, are not orthogonal and do not preserve $\mathrm{Tr}\,\boldsymbol{\sigma}$ (they belong to the non-compact subgroup of $Sp_{(2N,\mathbb{R})}$). This mathematical difference between squeezers and phase-space rotations accounts, in a quite elegant way, for the difference between *active* (energy non-preserving) and passive (energy preserving) optical transformations [24].

Let us remark that *local* symplectic operations belong to the group $\mathrm{Sp}(2,\mathbb{R})^{\oplus N}$. They correspond, at the Hilbert space level, to tensor products of unitary transformations, each acting on the state space of a single mode. It is useful to note that the determinants of each 2×2 submatrix of an N-mode CM, equation (19), are *all* invariants under local symplectic operations $S \in \mathrm{Sp}(2,\mathbb{R})^{\oplus N}$. This mathematical property reflects the physical requirement that marginal informational properties and correlations between individual subsystems cannot be altered by local operations alone.

3.2.1. Symplectic eigenvalues and invariants. A symplectic transformation of major importance is the one that diagonalizes a Gaussian state in the basis of normal modes. Through this decomposition, thanks to Williamson theorem [25], the CM of an N-mode Gaussian state can always be written in the so-called Williamson normal or diagonal form

$$\boldsymbol{\sigma} = S^{\mathsf{T}} \boldsymbol{\nu} S, \tag{27}$$

where $S \in Sp_{(2N,\mathbb{R})}$ and $\boldsymbol{\nu}$ is the CM

$$\boldsymbol{\nu} = \bigoplus_{k=1}^{N} \begin{pmatrix} \nu_k & 0 \\ 0 & \nu_k \end{pmatrix}, \tag{28}$$

corresponding to a tensor product state with a diagonal density matrix $\varrho^{\otimes}$ given by

$$\varrho^{\otimes} = \bigotimes_k \frac{2}{\nu_k + 1} \sum_{n=0}^{\infty} \left(\frac{\nu_k - 1}{\nu_k + 1} \right) |n\rangle_{kk}\langle n|, \tag{29}$$

where $|n\rangle_k$ denotes the number state of order n in the Fock space $\mathcal{H}_k$. In the Williamson form, each mode with frequency ω_k is a Gaussian state in thermal equilibrium at a temperature T_k, characterized by a Bose–Einstein statistical distribution of the thermal photons n_k, with average

$$\bar{n}_k = \frac{\nu_k - 1}{2} = \frac{1}{\exp\left(\frac{\hbar\omega_k}{k_B T_k}\right) - 1}. \tag{30}$$

The N quantities ν_k's form the *symplectic spectrum* of the CM $\boldsymbol{\sigma}$ and are invariant under the action of global symplectic transformations on the matrix $\boldsymbol{\sigma}$. The symplectic eigenvalues can be computed as the orthogonal eigenvalues of the matrix $|\mathrm{i}\Omega\boldsymbol{\sigma}|$ [26] and are thus determined by N invariants of the characteristic polynomial of such a matrix [21]. One global symplectic invariant is simply the determinant of the CM (whose invariance is a consequence of the fact that $\mathrm{Det}\, S = 1 \forall S \in Sp_{(2N,\mathbb{R})}$), which, once computed in the Williamson diagonal form, reads

$$\mathrm{Det}\,\boldsymbol{\sigma} = \prod_{k=1}^{N} \nu_k^2. \tag{31}$$

Another important invariant under global symplectic operations is the so-called *seralian* Δ [27], defined as the sum of the determinants of all 2×2 submatrices of a CM $\boldsymbol{\sigma}$, equation (19), which can be readily computed in terms of its symplectic eigenvalues as

$$\Delta(\boldsymbol{\sigma}) = \sum_{k=1}^{N} \nu_k^2. \tag{32}$$

The invariance of $\Delta_{\boldsymbol{\sigma}}$ in the multimode case [21] follows from its invariance in the case of two-mode states, proved in [28], and from the fact that any symplectic transformation can be decomposed as the product of two-mode transformations [29].

3.2.2. Symplectic representation of the uncertainty principle. The symplectic eigenvalues ν_k provide a powerful tool to access essential information on the properties of the Gaussian state $\boldsymbol{\sigma}$ [28]. For instance, let us consider the uncertainty relation (18). Since the inverse of a symplectic operation is itself symplectic, one has from equation (22), $S^{-1^{\mathsf{T}}}\Omega S^{-1} = \Omega$, so that in equation (18) is *equivalent* to $\boldsymbol{\nu} + \mathrm{i}\Omega \geqslant 0$. In terms of the symplectic eigenvalues ν_k, the uncertainty relation then simply reads

$$\nu_k \geqslant 1. \tag{33}$$

Inequality (33) is completely equivalent to the uncertainty relation (18) provided that the CM $\boldsymbol{\sigma}$ satisfies $\boldsymbol{\sigma} \geqslant 0$.

Without loss of generality, one can rearrange the modes of an N-mode state such that the corresponding symplectic eigenvalues are sorted in ascending order:

$$\nu_- \equiv \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{N-1} \leqslant \nu_N \equiv \nu_+.$$

With this notation, the uncertainty relation reduces to $\nu_1 \geqslant 1$. We remark that the full saturation of the uncertainty principle can only be achieved by *pure* N-mode Gaussian states, for which

$$\nu_i = 1 \qquad \forall i = 1, \ldots, N,$$

meaning that the Williamson normal form of any pure Gaussian state is the vacuum $|0\rangle$ of the N-mode Hilbert space $\mathcal{H}$. Instead, mixed states such that $\nu_{i\leqslant k} = 1$ and $\nu_{i>k} > 1$, with $1 \leqslant k \leqslant N$, only partially saturate the uncertainty principle, with partial saturation becoming weaker with decreasing k. Such states are minimum-uncertainty mixed Gaussian states in

Table 2. Schematic comparison between Hilbert-space and phase-space pictures for N-mode Gaussian states. The first two rows are taken from table 1 and apply to general states of CV systems. The following rows are special to Gaussian states, relying on the covariance matrix description and the properties of the symplectic group.

	Hilbert space $\mathcal{H}$	Phase space Γ
Dimension	∞	$2N$
Structure	$\bigotimes$	$\bigoplus$
Description	ϱ	σ
Bona fide	$\varrho \geqslant 0$	$\sigma + \mathrm{i}\Omega \geqslant 0$
Operations	$\underset{\varrho \mapsto U \varrho U^\dagger}{U : U^\dagger U = \mathbb{1}}$	$\underset{\sigma \mapsto S \sigma S^\mathsf{T}}{S : S^\mathsf{T} \Omega S = \Omega}$
Spectra	$\underset{0 \leqslant \lambda_k \leqslant 1}{U \varrho U^\dagger = \mathrm{diag}\{\lambda_k\}}$	$\underset{1 \leqslant \nu_k < \infty}{S \sigma S^\mathsf{T} = \mathrm{diag}\{\nu_k\}}$
Pure states	$\lambda_i = 1, \lambda_{j \neq i} = 0$	$\nu_j = 1, \forall j = 1, \ldots, N$
Purity	$\mathrm{Tr}\, \varrho^2 = \sum_k \lambda_k^2$	$1/\sqrt{\mathrm{Det}\, \sigma} = \prod_k \nu_k^{-1}$

the sense that the phase quadrature operators of the first k modes satisfy the Robertson–Schrödinger minimum uncertainty, while, for the remaining $N - k$ modes, the state indeed contains some additional thermal correlations which are responsible for the global mixedness of the state.

The *symplectic rank* $\aleph$ of a CM $\boldsymbol{\sigma}$ is defined as the number of its symplectic eigenvalues different from 1, corresponding to the number of non-vacua normal modes [30]. A Gaussian state is pure if and only if $\aleph = 0$. For mixed N-mode states one has $1 \leqslant \aleph \leqslant N$ depending on their degree of partial minimum-uncertainty saturation. This is in analogy with the standard rank of finite-dimensional (density) matrices, defined as the number of nonvanishing eigenvalues; in that case, only pure states $\varrho = |\psi\rangle\langle\psi|$ have rank 1, and mixed states have in general higher rank. As we will now show, *all* the informational properties of Gaussian states can be recast in terms of their symplectic spectra.

A mnemonic summary of the main structural features of Gaussian states in the phase-space/CM description (including the definition of purity given in the next subsection) is provided in table 2.

3.3. Degree of information contained in a Gaussian state

3.3.1. Measures of information. The degree of *information* contained in a quantum state corresponds to the amount of knowledge that we possess *a priori* on predicting the outcome of any test performed on the state [31].

The simplest measure of such information is the *purity* of a quantum state ϱ:

$$\mu(\varrho) = \mathrm{Tr}\, \varrho^2. \tag{34}$$

For states belonging to a given Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = D$, the purity varies in the range

$$\frac{1}{D} \leqslant \mu \leqslant 1.$$

The minimum is reached by the totally random mixture; the upper bound is saturated by pure states. In the limit of CV systems ($D \to \infty$), the minimum purity tends asymptotically to zero. Accordingly, the 'impurity' or degree of *mixedness* of a quantum state ϱ, which characterizes our ignorance before performing any quantum test on ϱ, can be quantified by the functional

$$S_L(\varrho) = \frac{N}{N-1}(1 - \mu) = \frac{N}{N-1}(1 - \mathrm{Tr}\, \varrho^2). \tag{35}$$

The *linear entropy* S_L (ranging between 0 and 1) defined by equation (35) is a very useful measure of mixedness in quantum mechanics and quantum information theory due to its direct connection with the purity and its computational simplicity.

In general, the degree of mixedness of a quantum state ϱ can be characterized completely by the knowledge of all the associated Schatten p-norms [32]

$$\|\varrho\|_p = (\mathrm{Tr}|\varrho|^p)^{\frac{1}{p}} = (\mathrm{Tr}\,\varrho^p)^{\frac{1}{p}}, \qquad \text{with} \quad p \geqslant 1. \tag{36}$$

In particular, the case $p = 2$ is directly related to the purity μ, equation (34), as it is essentially equivalent (up to a normalization) to the linear entropy, equation (35). The p-norms are multiplicative on tensor product states and thus determine a family of non-extensive 'generalized entropies' S_p [33, 34], defined as

$$S_p(\varrho) = \frac{1 - \mathrm{Tr}\,\varrho^p}{p - 1}, \qquad p > 1. \tag{37}$$

The generalized entropies S_p's range from 0 for pure states to $1/(p-1)$ for completely mixed states with fully degenerate eigenspectra. We also note that, for any given quantum state, S_p is a monotonically decreasing function of p. Finally, another important class of entropic measures includes the Rényi entropies [35]

$$S_p^R(\varrho) = \frac{\log \mathrm{Tr}\,\varrho^p}{1 - p}, \qquad p > 1. \tag{38}$$

It can be shown that [26]

$$\lim_{p \to 1+} S_p(\varrho) = \lim_{p \to 1+} S_p^R(\varrho) = -\mathrm{Tr}(\varrho \log \varrho). \tag{39}$$

Therefore, even the *von Neumann entropy*

$$S_V(\varrho) = -\mathrm{Tr}\varrho \log \varrho \tag{40}$$

can be defined, in terms of p-norms, as a suitable limit within the framework of generalized entropies.

The von Neumann entropy is *subadditive* [36]. Consider a bipartite system $\mathcal{S}$ (described by the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$) in the state ϱ. Then,

$$S_V(\varrho) \leqslant S_V(\varrho_1) + S_V(\varrho_2), \tag{41}$$

where $\varrho_{1,2}$ are the reduced density matrices $\varrho_{1,2} = \mathrm{Tr}_{2,1}\varrho$ associated with subsystems $\mathcal{S}_{1,2}$. For states of the form $\varrho^{\otimes} = \varrho_1 \otimes \varrho_2$, equation (41) is saturated, yielding that the von Neumann entropy is *additive* on tensor product states:

$$S_V(\varrho_1 \otimes \varrho_2) = S_V(\varrho_1) + S_V(\varrho_2). \tag{42}$$

The purity, equation (34), is instead *multiplicative* on product states, as the trace of a product equates the product of the traces:

$$\mu(\varrho_1 \otimes \varrho_2) = \mu(\varrho_1) \cdot \mu(\varrho_2). \tag{43}$$

Considering a joint classical probability distribution over the variables X and Y, one has for the Shannon entropy,

$$S(X, Y) \geqslant S(X), S(Y). \tag{44}$$

The Shannon entropy of a joint probability distribution is always greater than the Shannon entropy of each marginal probability distribution, meaning that there is less information in a global classical system than in any of its parts. On the other hand, consider a bipartite quantum system in a pure state $\varrho = |\psi\rangle\langle\psi|$. We have then for the von Neumann entropies: $S_V(\varrho) = 0$, $S_V(\varrho_1) = S_V(\varrho_2) \geqslant 0$. The global state ϱ has been prepared in a well-defined way, but if we measure local observables on the subsystems the measurement outcomes are unavoidably affected by random noise. One cannot reconstruct the whole information about how the global system was prepared in the state ϱ (apart from the trivial instance of ϱ being a product state $\varrho = \varrho_1 \otimes \varrho_2$), by only looking separately at the two subsystems. Information is rather encoded in nonlocal and nonfactorizable quantum correlations—*entanglement*—between the two subsystems. This highlights the fundamental difference between classical and quantum distributions of information.

3.3.2. Entropic measures for Gaussian states. We now illustrate how to evaluate the measures of information (or lack thereof) defined above for Gaussian states [37].

The generalized purities $\mathrm{Tr}\,\varrho^p$ defined by equation (36) are invariant under global unitary operations. Therefore, for any N-mode Gaussian state they are only functions of the symplectic eigenvalues ν_k of $\boldsymbol{\sigma}$. In fact, a symplectic transformation acting on $\boldsymbol{\sigma}$ is embodied by a unitary (trace-preserving) operator acting on ϱ, so that $\mathrm{Tr}\,\varrho^p$ can be easily computed on the Williamson diagonal state $\boldsymbol{\nu}$ of equation (28). One obtains [37]

$$\mathrm{Tr}\,\varrho^p = \prod_{i=1}^{N} g_p(\nu_i), \tag{45}$$

where

$$g_p(x) = \frac{2^p}{(x+1)^p - (x-1)^p}.$$

A remarkable consequence of equation (45) is that

$$\mu(\varrho) = \frac{1}{\prod_i \nu_i} = \frac{1}{\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}}. \tag{46}$$

Regardless of the number of modes, the purity of a Gaussian state is *fully determined* by the global symplectic invariant $\mathrm{Det}\,\boldsymbol{\sigma}$ alone, equation (31). We recall that the purity is related to the linear entropy S_L via equation (35), which in CV systems simply becomes $S_L = 1 - \mu$. A second consequence of equation (45) is that, together with equations (37) and (39), it allows for the computation of the von Neumann entropy S_V, equation (40), of a Gaussian state ϱ, yielding [28]

$$S_V(\varrho) = \sum_{i=1}^{N} f(\nu_i), \tag{47}$$

where

$$f(x) \equiv \frac{x+1}{2}\log\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log\left(\frac{x-1}{2}\right). \tag{48}$$

Such an expression for the von Neumann entropy of a Gaussian state was first explicitly given in [38]. Note that S_V diverges on infinitely mixed CV states, while S_L is normalized to 1. Let us remark that, clearly, the symplectic spectrum of single-mode Gaussian states, which consists of only one eigenvalue ν_1, is fully determined by the invariant $\mathrm{Det}\,\boldsymbol{\sigma} = \nu_1^2$. Therefore, all the entropies S_p's (and S_V as well) are just increasing functions of $\mathrm{Det}\,\boldsymbol{\sigma}$ (i.e. of S_L) and induce *the same* hierarchy of mixedness on the set of one-mode Gaussian states. This is no longer true for multimode states, even in the simplest instance of two-mode states [37].

3.4. Standard forms of special Gaussian states

The symplectic analysis applied on Gaussian CMs implies that some of them can be suitably reduced under local operations. Such reductions go under the name of *standard forms*.

3.4.1. Pure states: phase-space Schmidt decomposition. In general, *pure* Gaussian states of a bipartite CV system admit a physically insightful decomposition at the CM level [39–41], which can be regarded as the direct analogue of the Schmidt decomposition for pure discrete-variable states [42]. Let us recall what happens in the finite-dimensional case. With respect to a bipartition of a pure state $|\psi\rangle_{A|B}$ into two subsystems $\mathcal{S}_A$ and $\mathcal{S}_B$, one can diagonalize (via an operation $U_A \otimes U_B$ which is local unitary according to the considered bipartition) the two reduced density matrices $\varrho_{A,B}$, to find that they have the same rank and exactly the same nonzero eigenvalues $\{\lambda_k\}$ ($k = 1, \ldots, \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}$). The reduced state of the higher dimensional subsystem (say $\mathcal{S}_B$) will accommodate $(\dim \mathcal{H}_b - \dim \mathcal{H}_A)$ additional 0's in its spectrum. The state $|\psi\rangle_{A|B}$ takes thus the Schmidt form $|\psi\rangle = \sum_{k=1}^{d} \lambda_k |u_k, v_k\rangle$.

Looking at the mapping provided in table 2, one can deduce what happens for Gaussian states. Given a Gaussian CM $\boldsymbol{\sigma}_{A|B}$ of an arbitrary number N of modes, where subsystem $\mathcal{S}_A$ comprises N_A modes and subsystem $\mathcal{S}_B N_B$ modes (with $N_A + N_B = N$), then one can perform the Williamson decomposition, equation (27), in both reduced CMs (via a local symplectic operation $S_A \oplus S_B$), to find that they have the same symplectic rank and the same non-unit symplectic eigenvalues $\{\nu_k\}$ ($k = 1, \ldots, \min\{N_A, N_B\}$). The reduced state of the higher dimensional subsystem (say $\mathcal{S}_B$) will accommodate $(N_B - N_A)$ additional 1's in its symplectic spectrum. With respect to an arbitrary $A|B$ bipartition, therefore, the CM $\boldsymbol{\sigma}^p$ of any pure N-mode Gaussian state is locally equivalent to the form $\boldsymbol{\sigma}^p_S = (S_A \oplus S_B)\boldsymbol{\sigma}^p(S_A \oplus S_B)^{\mathsf{T}}$, with

$$\boldsymbol{\sigma}^p_S = \begin{array}{c} \qquad\overbrace{\hphantom{C_1\ \diamond\ \diamond\ C_{N_A}}}^{N_A}\qquad\quad \overbrace{\hphantom{S_1\ \diamond\ \diamond\ S_{N_A}\ \diamond\ \diamond\ \diamond}}^{N_B} \\ \left(\begin{array}{cccc|ccccccc}
\boldsymbol{C}_1 & \diamond & \diamond & \diamond & \boldsymbol{S}_1 & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond \\
\diamond & \boldsymbol{C}_2 & \diamond & \diamond & \diamond & \boldsymbol{S}_2 & \diamond & \diamond & \diamond & \diamond & \diamond \\
\diamond & \diamond & \ddots & \diamond & \diamond & \diamond & \ddots & \diamond & \diamond & \diamond & \diamond \\
\diamond & \diamond & \diamond & \boldsymbol{C}_{N_A} & \diamond & \diamond & \diamond & \boldsymbol{S}_{N_A} & \diamond & \diamond & \diamond \\
\hline
\boldsymbol{S}_1 & \diamond & \diamond & \diamond & \boldsymbol{C}_1 & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond \\
\diamond & \boldsymbol{S}_2 & \diamond & \diamond & \diamond & \boldsymbol{C}_2 & \diamond & \diamond & \diamond & \diamond & \diamond \\
\diamond & \diamond & \ddots & \diamond & \diamond & \diamond & \ddots & \diamond & \diamond & \diamond & \diamond \\
\diamond & \diamond & \diamond & \boldsymbol{S}_{N_A} & \diamond & \diamond & \diamond & \boldsymbol{C}_{N_A} & \diamond & \diamond & \diamond \\
\diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \mathbb{1} & \diamond & \diamond \\
\diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \ddots & \diamond \\
\diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \mathbb{1}
\end{array}\right) \end{array}. \tag{49}$$

Here, each element denotes a 2×2 submatrix, in particular the diamonds ($\diamond$) correspond to null matrices, $\mathbb{1}$ to the identity matrix and

$$\boldsymbol{C}_k = \begin{pmatrix} \nu_k & 0 \\ 0 & \nu_k \end{pmatrix}, \qquad \boldsymbol{S}_k = \begin{pmatrix} \sqrt{\nu_k^2 - 1} & 0 \\ 0 & -\sqrt{\nu_k^2 - 1} \end{pmatrix}.$$

The matrices $\boldsymbol{C}_k$ contain the symplectic eigenvalues $\nu_k \neq 1$ of both reduced CMs. By expressing them in terms of hyperbolic functions, $\nu_k = \cosh(2r_k)$, and by comparison with equation (21), one finds that each two-mode CM

$$\begin{pmatrix} \boldsymbol{C}_k & \boldsymbol{S}_k \\ \boldsymbol{S}_k & \boldsymbol{C}_k \end{pmatrix},$$

encoding correlations between a single mode from $\mathcal{S}_A$ and a single mode from $\mathcal{S}_B$, is a two-mode squeezed state with squeezing r_k. Therefore, the Schmidt form of a pure N-mode

Gaussian state with respect to an $(N_A \times N_B)$-mode bipartition (with $N_B \geqslant N_A$) is that of a direct sum [39, 41]

$$\sigma_S^p = \bigoplus_{i=1}^{N_A} \sigma_{i,j}^{\mathrm{sq}}(r_i) \bigoplus_{k=2N_A+1}^{N} \sigma_k^0, \tag{50}$$

where mode $i \in \mathcal{S}_A$, mode $j \equiv i+N_A \in \mathcal{S}_B$ and $\sigma_k^0 = \mathbb{1}_2$ is the CM of the vacuum state of mode $k \in \mathcal{S}_B$. This corresponds, on the Hilbert space level, to the product of two-mode squeezed states, tensor additional uncorrelated vacuum modes in the higher dimensional subsystem ($\mathcal{S}_B$ in our notation) [40]. The phase-space Schmidt decomposition is a very useful tool both for the understanding of the structural features of Gaussian states in the CM formalism and for the evaluation of their entanglement properties. Note that the validity of such a decomposition can be extended to mixed states with a fully degenerate symplectic spectrum, i.e. a Williamson normal form proportional to the identity [40, 41]. As a straightforward consequence of equation (50), any pure two-mode Gaussian state is equivalent, up to local unitary operations, to a two-mode squeezed state of the form (21).

We will now show that for (generally mixed) Gaussian states with some local symmetry constraints, a similar phase-space reduction is available, such that multimode properties (like entanglement) can be unitarily reduced to two-mode ones.

3.4.2. Symmetric and bisymmetric states. Very often in quantum information, and in particular in the theory of entanglement, peculiar roles are played by *symmetric states*, that is, states that are either invariant under a particular group of transformations—like Werner states of qu*d*its [43]—or under permutation of two or more parties in a multipartite system, such as ground and thermal states of translationally invariant Hamiltonians (e.g. of harmonic lattices) [44]. Here, we will introduce classes of Gaussian states globally invariant under all permutation of the modes (fully symmetric states) or locally invariant in each of the two subsystems across a global bipartition of the modes (bisymmetric states). The properties of their symplectic spectrum allow us to explicitly determine a standard form in both cases. Here, we will limit ourselves to list and collect the results that will be useful for the computation and exploitation of entanglement in the corresponding states. The detailed discussions and all the rigorous proofs can be found in [45]. Unless explicitly stated, in the following we will be dealing with generally mixed states.

A multimode Gaussian state ϱ is 'fully symmetric' if it is invariant under the exchange of any two modes. In the following, we will consider the fully symmetric M-mode and N-mode Gaussian states ϱ_{α^M} and ϱ_{β^N}, with CMs σ_{α^M} and σ_{β^N}. Due to symmetry, we have that

$$\sigma_{\alpha^M} = \begin{pmatrix} \alpha & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \alpha & \varepsilon & \vdots \\ \vdots & \varepsilon & \ddots & \varepsilon \\ \varepsilon & \cdots & \varepsilon & \alpha \end{pmatrix}, \qquad \sigma_{\beta^N} = \begin{pmatrix} \beta & \zeta & \cdots & \zeta \\ \zeta & \beta & \zeta & \vdots \\ \vdots & \zeta & \ddots & \zeta \\ \zeta & \cdots & \zeta & \beta \end{pmatrix}, \tag{51}$$

where α, ε, β and ζ are 2×2 real symmetric submatrices (the symmetry of ε and ζ stems again from the symmetry under the exchange of any two modes).

A fully symmetric N-mode Gaussian states with CM σ_{β^N} admits the following standard form [46, 45]. The 2×2 blocks β and ζ of σ_{β^N}, defined by equation (51), can be brought by means of local, single-mode symplectic operations $S \in Sp_{(2,\mathbb{R})}^{\oplus N}$ into the form $\beta = \mathrm{diag}(b, b)$ and $\zeta = \mathrm{diag}(z_1, z_2)$. Obviously, analogous results hold for the M-mode CM σ_{α^M} of equation (51), whose 2×2 submatrices can be brought to the form $\boldsymbol{\alpha} = \mathrm{diag}(a, a)$ and $\boldsymbol{\varepsilon} = \mathrm{diag}(e_1, e_2)$.

Let us now generalize this analysis to the $(M + N)$-mode Gaussian states with CM $\boldsymbol{\sigma}$, which results from a correlated combination of the fully symmetric blocks $\boldsymbol{\sigma}_{\boldsymbol{\alpha}^M}$ and $\boldsymbol{\sigma}_{\boldsymbol{\beta}^N}$:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{\boldsymbol{\alpha}^M} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^{\mathsf{T}} & \boldsymbol{\sigma}_{\boldsymbol{\beta}^N} \end{pmatrix}, \tag{52}$$

where $\boldsymbol{\Gamma}$ is a $2M \times 2N$ real matrix formed by identical 2×2 blocks $\boldsymbol{\gamma}$. Clearly, $\boldsymbol{\Gamma}$ is responsible for the correlations existing between the M-mode and the N-mode parties. Once again, the identity of the submatrices $\boldsymbol{\gamma}$ is a consequence of the local invariance under mode exchange, internal to the M-mode and N-mode parties. States of the form of equation (52) will be henceforth referred to as 'bisymmetric' [46, 45]. By evaluating the symplectic spectrum of such a bisymmetric state, and by comparing it with the spectra of the reduced fully symmetric CMs, one can show the following central result [45], which applies to all (generally mixed) bisymmetric Gaussian states, and is somehow analogous to—but independent of—the phase-space Schmidt decomposition of pure Gaussian states (and of mixed states with fully degenerate symplectic spectrum). *The bisymmetric $(M + N)$-mode Gaussian state with CM $\boldsymbol{\sigma}$, equation (52), can be brought, by means of a local unitary (symplectic) operation with respect to the $M \times N$ bipartition with reduced CMs $\boldsymbol{\sigma}_{\boldsymbol{\alpha}^M}$ and $\boldsymbol{\sigma}_{\boldsymbol{\beta}^N}$, to a tensor product of single-mode uncorrelated states and of a two-mode Gaussian state comprised of one mode from the M-mode block and one mode from the N-mode block.*

We will explore the consequences of such a unitary localization on the entanglement properties of bisymmetric states in the next sections. Let us just note that fully symmetric Gaussian states, equation (51), are special instances of bisymmetric states with respect to any global bipartition of the modes.

4. Theory of bipartite entanglement for Gaussian states

4.1. Entanglement and nonlocality

When two quantum systems come to interact, some quantum correlation is established between the two of them. This correlation persists even when the interaction is switched off and the two systems are spatially separated[5]. If we measure a local observable on the first system, its state collapses in an eigenstate of that observable. In the ideal case of no environmental decoherence, the state of the second system is instantly modified. The question then arises about what is the mechanism responsible for this 'spooky action at a distance' [22].

Suppose we have a bipartite or multipartite quantum state. An apparently innocent question like

Does this state contain quantum correlations?

turns out to be very hard to answer [48–50]. The first step towards a solution concerns a basic understanding of what such a question really means.

One may argue that a system contains quantum correlations if the observables associated with the different subsystems are correlated and their correlations cannot be reproduced with purely classical means. This implies that some form of inseparability or nonfactorizability is necessary to properly take into account those correlations. For what concerns globally *pure states* of the composite quantum system, it is relatively easy to control if the correlations are

[5] Entanglement can also be created without direct interaction between the subsystems, via the common interaction with other *assisting* parties. This and related mechanisms define the so-called entanglement swapping [47].

of genuine quantum nature. In particular, it is enough to check if a Bell-CHSH inequality [51, 52] is violated [53], to conclude that a pure quantum state is entangled. There are in fact many different criteria to characterize entanglement, but all of them are practically based on equivalent forms of nonlocality in pure quantum states.

This scheme fails with *mixed states*. Contrary to a pure state, a statistical mixture can be prepared in (generally infinitely) many different ways. Being it impossible to reconstruct the original preparation of the state, one cannot extract all the information it contains. Accordingly, there is no completely general and practical criterion to decide whether correlations in a mixed quantum state are of classical or quantum nature. Moreover, different manifestations of quantum inseparability are in general *not* equivalent. For instance, one pays more (in units of singlets $(|00\rangle + |11\rangle)/\sqrt{2})$ to create an entangled mixed state ϱ (entanglement cost [54]) than to reconvert ϱ into a product of singlets (distillable entanglement [54]) via local operations and classical communication (LOCC) [55]. An important example has been introduced by Werner [43], who defined a parametric family of mixed states (known as Werner states) which, in some range of the parameters, are entangled (inseparable) without violating any Bell inequality on local realistic models. These states thus admit a description in terms of local hidden variables. It is indeed an open question in quantum information theory to prove whether any entangled state violates some kind of Bell-type inequality [56, 57].

In fact, entanglement and nonlocality are different resources from an operational point of view [58]. This can be understood within the general framework of no-signalling theories which exhibit stronger nonlocal features than those of quantum mechanics. Let us briefly recall what is intended by nonlocality according to Bell [59]: there exists in nature a channel that allows one to distribute correlations between distant observers, such that the correlations are not already established at the source, and the correlated random variables can be created in a configuration of space-like separation, i.e. no normal signal (assuming no superluminal transmission) can be the cause of the correlations [22, 51]. A convenient understanding of the phenomenon of nonlocality is then given by quantum mechanics, which describes the channel as a pair of entangled particles. But such interpretation is not the only possible one. In recent years, there has been a growing interest in constructing alternative descriptions of this channel, mainly assuming a form of communication [60], or the use of a hypothetical 'nonlocal machine' [61] able to violate the CHSH inequality [52] up to its algebraic value of 4 (while the local realism threshold is 2 and the maximal violation admitted by quantum mechanics is $2\sqrt{2}$, the Cirel'son bound [62]). Looking at these alternative descriptions allows us to quantify how powerful quantum mechanics is by comparing its performance with that of other resources [63].

4.2. *Entanglement of pure states*

It is now well understood that entanglement in a pure bipartite quantum state $|\psi\rangle$ is equivalent to the degree of mixedness of each subsystem. Accordingly, it can be properly quantified by the *entropy of entanglement* $E_V(|\psi\rangle)$, defined as the von Neumann entropy, equation (40), of the reduced density matrices [64],

$$E_V(|\psi\rangle) = S_V(\varrho_1) = S_V(\varrho_2) = -\sum_{k=1}^{d} \lambda_k^2 \log \lambda_k^2. \tag{53}$$

The entropy of entanglement is by definition invariant under local unitary operations

$$E_V((\hat{U}_1 \otimes \hat{U}_2)|\psi\rangle) = E_V(|\psi\rangle), \tag{54}$$

and it can be shown [65] that $E_V(|\psi\rangle)$ cannot increase under LOCC performed on the state $|\psi\rangle$: this is a fundamental physical requirement as it reflects the fact that entanglement cannot be created via LOCC only [66, 67]. It can be formalized as follows. Let us suppose, starting with a state $|\psi\rangle$ of the global system $\mathcal{S}$, to perform local measurements on $\mathcal{S}_1$ and $\mathcal{S}_2$, and to obtain, after the measurement, the state $|\varphi_1\rangle$ with probability p_1, the state $|\varphi_2\rangle$ with probability p_2 and so on. Then

$$E_V(|\psi\rangle) \geqslant \sum_k p_k E_V(|\varphi_k\rangle). \tag{55}$$

Note that entanglement cannot increase *on average*, that is nothing prevents, for a given k, that $E_V(|\varphi_k\rangle) > E_V(|\psi\rangle)$. The concept of *entanglement distillation* is based on this fact [54, 68, 69]: with a probability p_k, it is possible to increase entanglement via LOCC manipulations.

4.3. Entanglement of mixed states

A mixed state ϱ can be decomposed as a convex combination of pure states,

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|. \tag{56}$$

equation (56) tells us how to create the state described by the density matrix ϱ: we have to prepare the state $|\psi_1\rangle$ with probability p_1, the state $|\psi_2\rangle$ with probability p_2, etc. For instance, we could collect N copies ($N \gg 1$) of the system, prepare $n_k \simeq N p_k$ of them in the state $|\psi_k\rangle$ and pick a copy at random.

The difficulty lies in the fact that the decomposition of equation (56) is not unique: apart from pure states, there exist infinitely many decompositions of a generic ϱ, meaning that the mixed state can be prepared in infinitely many different ways. This has important consequences on the determination of mixed-state entanglement. Suppose that for a bipartite system in a mixed state we detect, by local measurements, the presence of correlations between the two subsystems. Given the ambiguity on the state preparation, we cannot know *a priori* if those correlations arose from a quantum interaction between the subsystems (meaning entanglement) or were induced by means of LOCC (meaning classical correlations). It is thus clear that a mixed state can be defined separable (classically correlated) if there exists at least one way of engineering it by LOCC; on the other hand, it is entangled (quantumly correlated) if, among the infinite possible procedures for its preparation, there is no one which relies on LOCC alone [43].

However, deciding separability according to the above definition would imply checking all the infinitely many decompositions of a state ϱ and looking for at least one, expressed as a convex combination of product states, to conclude that the state is not entangled. This is clearly impractical. For this reason, several *operational* criteria have been developed in order to detect entanglement in mixed quantum states [70–72]. Some of them, of special relevance to Gaussian states of CV systems, are discussed in the following.

4.4. Separability and distillability of Gaussian states

4.4.1. PPT criterion. One of the most powerful results to date in the context of separability criteria is the Peres–Horodecki condition [73, 74]. It is based on the operation of *partial transposition* of the density matrix of a bipartite system, obtained by performing transposition with respect to the degrees of freedom of one subsystem only. Peres criterion states that if a state ϱ_s is separable then its partial transpose $\varrho_s^{\mathsf{T}_1}$ (with respect, e.g., to subsystem $\mathcal{S}_1$) is a valid density matrix, in particular positive semidefinite, $\varrho_s^{\mathsf{T}_1} \geqslant 0$. Obviously, the same holds for $\varrho_s^{\mathsf{T}_2}$. Positivity of the partial transpose (PPT) is therefore a necessary condition for separability [73]. The converse (i.e. $\varrho^{\mathsf{T}_1} \geqslant 0 \Rightarrow \varrho$ separable) is in general false, but is has been proven true for low-dimensional systems, specifically bipartite systems with Hilbert state space of dimensionality 2×2 and 2×3. In these cases, the PPT property is equivalent to separability [74]. For higher dimensional tensor product structures of Hilbert spaces, PPT entangled states (with $\varrho^{\mathsf{T}_1} \geqslant 0$) have been shown to exist [75]. These states are known as *bound entangled* [76], because their entanglement cannot be distilled to obtain maximally entangled states. The existence of bound entangled (undistillable) states with negative partial transposition has been conjectured [77, 78], but at present there is not yet evidence of this property [56].

Recently, the PPT criterion has been generalized to continuous-variable systems by Simon [13], who showed how the operation of transposition acquires in infinite-dimensional Hilbert spaces an elegant geometric interpretation in terms of time inversion (mirror reflection of the momentum operator). The PPT criterion is necessary and sufficient for the separability of all $(1 \times N)$-mode Gaussian states [13, 79, 80]. Under partial transposition, the CM $\boldsymbol{\sigma}_{A|B}$, where subsystem $\mathcal{S}_A$ groups N_A modes, and subsystem $\mathcal{S}_B$ is formed by N_B modes, is transformed into a new matrix

$$\tilde{\boldsymbol{\sigma}}_{A|B} \equiv \boldsymbol{\theta}_{A|B}\boldsymbol{\sigma}_{A|B}\boldsymbol{\theta}_{A|B}, \tag{57}$$

with

$$\boldsymbol{\theta}_{A|B} = \mathrm{diag}\{\underbrace{1, -1, 1, -1, \ldots, 1, -1}_{2N_A}, \underbrace{1, 1, 1, 1, \ldots, 1, 1}_{2N_B}\}.$$

Referring to the notation of equation (19), the partially transposed matrix $\tilde{\boldsymbol{\sigma}}_{A|B}$ differs from $\boldsymbol{\sigma}_{A|B}$ by a sign flip in the determinants of the intermodal correlation matrices, $\mathrm{Det}\,\varepsilon_{ij}$, with modes $i \in \mathcal{S}_A$ and modes $j \in s_B$.

The PPT criterion yields that a Gaussian state $\boldsymbol{\sigma}_{A|B}$ (with $N_A = 1$ and N_B arbitrary) is *separable* if and only if the partially transposed $\tilde{\boldsymbol{\sigma}}_{A|B}$ is a *bona fide* CM, that is it satisfies the uncertainty principle equation (18),

$$\tilde{\boldsymbol{\sigma}}_{A|B} + \mathrm{i}\Omega \geqslant 0. \tag{58}$$

This property in turn reflects the positivity of the partially transposed density matrix ϱ^{T_A} associated with the state ϱ. For Gaussian states with $N_A > 1$, not endowed with special symmetry constraints, the PPT condition is only necessary for separability, as bound entangled Gaussian states, whose entanglement is undistillable, exist already in the instance $N_A = N_B = 2$ [80].

We have demonstrated the existence of 'bisymmetric' $(N_A + N_B)$-mode Gaussian states for which PPT is again equivalent to separability [45]. In view of the invariance of the PPT criterion under local unitary transformations, and considering the results of section 3.4.2 on the unitary localization of bisymmetric Gaussian states, it is immediate to verity that the following property holds [45]. *For generic $N_A \times N_B$-mode bipartitions, the positivity of the partial transpose (PPT) is a necessary and sufficient condition for the separability of bisymmetric $(N_A + N_B)$-mode mixed Gaussian states of the form equation (52). In the case of fully symmetric mixed Gaussian states, equation (51), of an arbitrary number of bosonic modes, PPT is equivalent to separability across any global bipartition of the modes.*

This statement generalizes to multimode bipartitions the equivalence between separability and PPT for $1 \times N$ bipartite Gaussian states [80]. In particular, it implies that no bisymmetric bound entangled Gaussian states can exist [80, 81] and all the $N_A \times N_B$ multimode block entanglement of such states is distillable. Moreover, it justifies the use of the negativity and the logarithmic negativity as measures of entanglement for this class of multimode Gaussian states (6).

Table 3. Schematic comparison between the conditions of existence and the conditions of separability for Gaussian states, as expressed in different representations. The second column qualifies the PPT condition, which is always implied by separability, and equivalent to it in $1 \times N$ and in $M \times N$ bisymmetric Gaussian states.

	Physical	Separable
Density matrix	$\varrho \geqslant 0$	$\varrho^{\mathsf{T}_A} \geqslant 0$
Covariance matrix	$\sigma + \mathrm{i}\Omega \geqslant 0$	$\tilde{\sigma} + \mathrm{i}\Omega \geqslant 0$
Symplectic spectrum	$\nu_k \geqslant 1$	$\tilde{\nu}_k \geqslant 1$

The PPT criterion has an elegant symplectic representation. The partially transposed matrix $\tilde{\boldsymbol{\sigma}}$ of any N-mode Gaussian CM is still a positive and symmetric matrix. As such, it admits a Williamson normal-mode decomposition [25], equation (27), of the form

$$\tilde{\boldsymbol{\sigma}} = S^{\mathsf{T}} \tilde{\boldsymbol{\nu}} S, \tag{59}$$

where $S \in Sp_{(2N,\mathbb{R})}$ and $\tilde{\boldsymbol{\nu}}$ is the CM

$$\tilde{\boldsymbol{\nu}} = \bigoplus_{k=1}^{N} \begin{pmatrix} \tilde{\nu}_k & 0 \\ 0 & \tilde{\nu}_k \end{pmatrix}. \tag{60}$$

The N quantities $\tilde{\nu}_k$'s are the symplectic eigenvalues of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$. The symplectic spectrum $\{\nu_k\}$ of $\boldsymbol{\sigma}$ encodes the structural and informational properties of a Gaussian state. The partially transposed spectrum $\{\tilde{\nu}_k\}$ encodes the qualitative (and, to some extent, quantitative—see section 4.5) characterization of entanglement in the state. The PPT condition (58), i.e. the uncertainty relation for $\tilde{\boldsymbol{\sigma}}$, can be equivalently recast in terms of the parameters $\tilde{\nu}_k$'s as

$$\tilde{\nu}_k \geqslant 1. \tag{61}$$

We can, without loss of generality, rearrange the modes of an N-mode state such that the corresponding symplectic eigenvalues of the partial transpose $\tilde{\boldsymbol{\sigma}}$ are sorted in ascending order,

$$\tilde{\nu}_- \equiv \tilde{\nu}_1 \leqslant \tilde{\nu}_2 \leqslant \cdots \leqslant \tilde{\nu}_{N-1} \leqslant \tilde{\nu}_N \equiv \tilde{\nu}_+,$$

in analogy to what done in section 3.2.2 for the spectrum of $\boldsymbol{\sigma}$. With this notation, the PPT criterion across an arbitrary bipartition reduces to $\tilde{\nu}_1 \geqslant 1$ for all separable Gaussian states. If $\tilde{\nu}_1 < 1$, the corresponding Gaussian state $\boldsymbol{\sigma}$ is entangled. The symplectic characterization of physical and PPT Gaussian states is summarized in table 3.

The distillability problem for Gaussian states has been solved in quite general terms [81]: the entanglement of any non-PPT bipartite Gaussian state is distillable by LOCC. However, we recall that this entanglement can be distilled only resorting to non-Gaussian LOCC [82], since distilling Gaussian states with Gaussian operations is impossible [83–85].

4.4.2. Additional criteria for separability. Let us briefly mention alternative characterizations of separability for Gaussian states.

For a general Gaussian state of any $N_A \times N_B$ bipartition, a necessary and sufficient condition states that a CM $\boldsymbol{\sigma}$ corresponds to a separable state if and only if there exists a pair of CMs $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$, relative to the subsystems $\mathcal{S}_A$ and s_B respectively, such that the following inequality holds [80], $\boldsymbol{\sigma} \geqslant \boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B$. This criterion is not very useful in practice. Alternatively, one can introduce an operational criterion based on iterative applications of a nonlinear map, that is independent of (and strictly stronger than) the PPT condition [86], and completely qualifies separability for all bipartite Gaussian states.

Another powerful tool establish that the separability of quantum states is given by the so-called *entanglement witnesses*. A state ϱ is entangled if and only if there exists a Hermitian operator $\hat{\mathcal{W}}$ such that $\mathrm{Tr}\,\hat{\mathcal{W}}\varrho < 0$ and $\mathrm{Tr}\,\hat{\mathcal{W}}\sigma \geqslant 0$ for any state $\hat{\sigma} \in \mathcal{D}$, where $\mathcal{D} \subset \mathcal{H}$ is the convex and compact subset of separable states [57, 74]. The operator $\hat{\mathcal{W}}$ is the *witness* responsible for detecting entanglement in the state ϱ. According to the Hahn–Banach theorem, given a convex and compact set $\mathcal{D}$ and given $\varrho \notin \mathcal{D}$, there exists a hyperplane which separates ϱ from $\mathcal{D}$. Optimal entanglement witnesses induce a hyperplane which is tangent to the set $\mathcal{D}$ [87]. A sharper detection of separability can be achieved by means of nonlinear entanglement witnesses, curved towards the set $\mathcal{D}$ [88]. A comprehensive characterization of linear and nonlinear entanglement witnesses is available for CV systems [89] and can be efficiently applied to the detection of separability in arbitrary Gaussian states, both in the bipartite and in the multipartite context.

Finally, several operational criteria have been developed, that are specially useful in experimental settings. They are based on the violation of inequalities involving combinations of variances of canonical operators, and their validity ranges from the two-mode [79] to the multimode setting [90].

4.5. Quantification of bipartite entanglement in Gaussian states

The question of the quantification of bipartite entanglement for general (pure *and* mixed) cannot be considered completely solved yet. We have witnessed a proliferation of entanglement measures, each one motivated by specific contexts in which quantum correlations play a central role, and accounting for different, and in some cases inequivalent, operational characterizations and orderings of entangled states. Detailed expositions on the subject can be found, e.g., in [50, 71, 91, 92].

The first natural generalization of the quantification of entanglement to mixed states is certainly the *entanglement of formation* $E_F(\varrho)$ [54], defined as the convex-roof extension [93] of the entropy of entanglement equation (53), i.e. the weighted average of pure-state entanglement,

$$E_F(\varrho) = \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k E_V(|\psi_k\rangle), \tag{62}$$

minimized over all decompositions of the mixed state $\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. This is clearly an optimization problem of formidable difficulty, and an explicit solution is known only for the mixed states of two qubits [94], and for highly symmetric states such as Werner states and isotropic states in arbitrary dimension [95, 96]. In CV systems, an explicit expression for the entanglement of formation is available only for symmetric, two-mode Gaussian states [97]. To date, the additivity of the entanglement of formation remains an open problem [56].

4.5.1. Negativities. An important class of entanglement monotones is defined by the negativities, which quantify the violation of the PPT criterion for separability (see section 4.4.1), i.e. how much the partial transposition of ϱ fails to be positive. The *negativity* $\mathcal{N}(\varrho)$ [98, 99] is defined as

$$\mathcal{N}(\varrho) = \frac{\|\varrho^{\mathsf{T}_i}\|_1 - 1}{2}, \tag{63}$$

where

$$\|\hat{O}\|_1 = \mathrm{Tr}\sqrt{\hat{O}^\dagger \hat{O}} \tag{64}$$

is the trace norm of the operator $\hat{O}$. The negativity has the advantage of being a computable measure of entanglement, being

$$\mathcal{N}(\varrho) = \max\left\{0, -\sum_k \lambda_k^-\right\}, \tag{65}$$

where $\{\lambda_k^-\}$'s are the negative eigenvalues of the partial transpose.

The negativity can be defined for CV systems as well [100], even though a related measure is more often used, the *logarithmic negativity* $E_{\mathcal{N}}(\varrho)$ [99, 100],

$$E_{\mathcal{N}}(\varrho) = \log\|\varrho^{\mathsf{T}_i}\|_1 = \log[1 + 2\mathcal{N}(\varrho)]. \tag{66}$$

The logarithmic negativity is additive and, despite not being convex, is a full entanglement monotone under LOCC [101]; it is an upper bound for the distillable entanglement [99], $E_{\mathcal{N}}(\varrho) \geqslant E_D(\varrho)$, and coincides with the entanglement cost under operations preserving the positivity of the partial transpose [102]. Both the negativity and the logarithmic negativity fail to be continuous in trace norm on infinite-dimensional Hilbert spaces; however, this problem can be circumvented by restricting to physical states of finite mean energy [103].

The great advantage of the negativities is that they are easily *computable* for general Gaussian states; they provide a proper quantification of entanglement in particular for arbitrary $1 \times N$ and bisymmetric $M \times N$ Gaussian states, directly quantifying the degree of violation of the necessary and sufficient PPT criterion for separability, equation (61). Following [26, 37, 46, 100], the negativity of a Gaussian state with CM $\boldsymbol{\sigma}$ is given by

$$\mathcal{N}(\boldsymbol{\sigma}) = \begin{cases} \dfrac{1}{2}\left(\displaystyle\prod_k \tilde{\nu}_k^{-1} - 1\right), & \text{for} \quad k : \tilde{\nu}_k < 1, \\ 0 & \text{if} \quad \tilde{\nu}_i \geqslant 1 \forall i. \end{cases} \tag{67}$$

Here, the set $\{\tilde{\nu}_k\}$ is constituted by the symplectic eigenvalues of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$. Accordingly, the logarithmic negativity reads

$$E_{\mathcal{N}}(\boldsymbol{\sigma}) = \begin{cases} -\displaystyle\sum_k \log \tilde{\nu}_k, & \text{for} \quad k : \tilde{\nu}_k < 1, \\ 0 & \text{if} \quad \tilde{\nu}_i \geqslant 1 \forall i. \end{cases} \tag{68}$$

The following lemma is quite useful for the interpretation and the computation of the negativities in Gaussian states [21]. In an $(N_A + N_B)$-mode Gaussian state with CM $\boldsymbol{\sigma}_{A|B}$, at most

$$N_{\min} \equiv \min\{N_A, N_B\} \tag{69}$$

symplectic eigenvalues $\tilde{\nu}_k$ of the partial transpose $\tilde{\boldsymbol{\sigma}}_{A|B}$ can violate the PPT inequality (61) with respect to an $N_A \times N_B$ bipartition. Thanks to this result, in all $1 \times N$ Gaussian states and in all bisymmetric $M \times N$ Gaussian states (whose symplectic spectra exhibit degeneracy, see section 3.4.2), the negativities are quantified in terms of the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM alone. For $\tilde{\nu}_- \geqslant 1$, the state is separable, otherwise it is entangled; the smaller the $\tilde{\nu}_-$ the more entangled is the corresponding Gaussian state. In the limit of a vanishing partially transposed symplectic eigenvalue, $\tilde{\nu}_- \to 0$, the negativities grow unboundedly. In the special instance of two-mode Gaussian states, such a result had been originally derived in [37, 104].

4.5.2. Gaussian convex-roof extended measures. It is possible to define a family of entanglement measures exclusively defined for Gaussian states of CV systems [105]. The formalism of *Gaussian entanglement measures* (Gaussian EMs) has been introduced in

[106] where the 'Gaussian entanglement of formation' has been defined and analysed. The framework developed in [106] is general and allows us to define generic Gaussian EMs of bipartite entanglement by applying the Gaussian convex roof, that is, the convex roof over pure Gaussian decompositions only, to any *bona fide* measure of bipartite entanglement defined for pure Gaussian states.

The original motivation for the introduction of Gaussian EMs stems from the fact that the entanglement of formation [54], defined by equation (62), involves a nontrivial minimization of the pure-state entropy of entanglement over convex decompositions of bipartite mixed Gaussian states in ensemble of pure states. These pure states may be, in principle, non-Gaussian states of CV systems, thus rendering the analytical solution of the optimization problem in equation (62) extremely difficult even in the simplest instance of one mode per party. Nevertheless, in the special subset of two-mode symmetric mixed Gaussian states, the optimal convex decomposition of equation (62) has been exactly determined and is realized in terms of pure Gaussian states [97]. Apart from that case (which will be discussed in section 5.2.2), the determination of the entanglement of formation of nonsymmetric two-mode Gaussian states (and more general Gaussian states) is an open problem in the theory of entanglement [56]. However, inspired by the results achieved on two-mode symmetric states, one can at first try to restrict the problem only to decompositions into pure Gaussian states. The resulting measure, named as Gaussian entanglement of formation in [106], is an upper bound to the true entanglement of formation and coincides with it for symmetric two-mode Gaussian states.

In general, we can define a Gaussian EM G_E as follows. For any pure Gaussian state ψ with CM $\boldsymbol{\sigma}^p$, one has

$$G_E(\boldsymbol{\sigma}^p) \equiv E(\psi), \tag{70}$$

where E can be *any* proper measure of entanglement of pure states, defined as a monotonically increasing function of the entropy of entanglement (i.e. the von Neumann entropy of the reduced density matrix of one party).

For any mixed Gaussian state ϱ with CM $\boldsymbol{\sigma}$, one has [106]

$$G_E(\boldsymbol{\sigma}) \equiv \inf_{\boldsymbol{\sigma}^p \leqslant \boldsymbol{\sigma}} G_E(\boldsymbol{\sigma}^p). \tag{71}$$

If the function E is taken to be exactly the entropy of entanglement, equation (53), then the corresponding Gaussian EM is known as *Gaussian entanglement of formation* [106]. Any Gaussian EM is an entanglement monotone under Gaussian LOCC. The proof given in section IV of [106] for the Gaussian entanglement of formation automatically extends to every Gaussian EM constructed via the Gaussian convex roof of any proper measure E of pure-state entanglement.

In general, the definition, equation (71), involves an optimization over all pure Gaussian states with CM $\boldsymbol{\sigma}^p$ smaller than the CM $\boldsymbol{\sigma}$ of the mixed state whose entanglement one wishes to compute. This corresponds to taking the Gaussian convex roof. Despite being a simpler optimization problem than that appearing in the definition, equation (62), of the true entanglement of formation, the Gaussian EMs cannot be expressed in a simple closed form, not even in the instance of (nonsymmetric) two-mode Gaussian states. Some recent results on this issue have been obtained in [105], and will be discussed in section 5.4, in relation with the problem of the ordering of quantum states according to different measures of entanglement.

5. Entanglement of two-mode Gaussian states

We now discuss the characterization of the prototypical entangled states of CV systems, i.e. the two-mode Gaussian states. This includes the explicit determination of the negativities and their relationship with global and marginal entropic measures [27, 37, 104], and the evaluation of the Gaussian measures of entanglement [105]. We will then compare the two families of measures according to the ordering that they establish on entangled states.

5.1. Symplectic parametrization of two-mode Gaussian states

To study entanglement and informational properties (like global and marginal entropies) of two-mode Gaussian states, we can consider without loss of generality states whose CM $\boldsymbol{\sigma}$ is in the $Sp_{(2,\mathbb{R})} \oplus Sp_{(2,\mathbb{R})}$-invariant standard form derived in [13, 79]. Let us recall it here for the sake of clarity,

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}. \tag{72}$$

For two-mode states, the uncertainty principle in equation (18) can be recast as a constraint on the $Sp_{(4,\mathbb{R})}$ invariants (invariants under global, two-mode symplectic operations) $\mathrm{Det}\,\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma}) = \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} + 2\,\mathrm{Det}\,\boldsymbol{\gamma}$ [28],

$$\Delta(\boldsymbol{\sigma}) \leqslant 1 + \mathrm{Det}\,\boldsymbol{\sigma}. \tag{73}$$

The symplectic eigenvalues of a two-mode Gaussian state will be denoted as ν_- and ν_+, with $\nu_- \leqslant \nu_+$, with the uncertainty relation (33) reducing to

$$\nu_- \geqslant 1. \tag{74}$$

A simple expression for the $\nu_\mp$ can be found in terms of the two $Sp_{(4,\mathbb{R})}$ invariants [28, 100] [37, 104]

$$2\nu_\mp^2 = \Delta(\boldsymbol{\sigma}) \mp \sqrt{\Delta^2(\boldsymbol{\sigma}) - 4\,\mathrm{Det}\,\boldsymbol{\sigma}}. \tag{75}$$

According to equation (72), two-mode Gaussian states can be classified in terms of their four standard-form covariances a, b, c_+ and c_-. It is relevant to provide a reparametrization of standard-form states in terms of symplectic invariants which admit a direct interpretation for generic Gaussian states [27, 37, 104]. Namely, the parameters of equation (72) can be determined in terms of the two local symplectic invariants

$$\mu_1 = (\mathrm{Det}\,\boldsymbol{\alpha})^{-1/2} = 1/a, \qquad \mu_2 = (\mathrm{Det}\,\boldsymbol{\beta})^{-1/2} = 1/b, \tag{76}$$

which are the marginal purities of the reduced single-mode states and of the two global symplectic invariants

$$\mu = (\mathrm{Det}\,\boldsymbol{\sigma})^{-1/2} = \left[\left(ab - c_+^2\right)\left(ab - c_-^2\right)\right]^{-1/2}, \qquad \Delta = a^2 + b^2 + 2c_+c_-, \tag{77}$$

which are, respectively, the global purity equation (46) and the seralian equation (32). Equations (76) and (77) can be inverted to give a physical parametrization of two-mode states in terms of the four independent parameters μ_1, μ_2, μ and Δ. This parametrization is particularly useful for the evaluation of entanglement [37, 104].

5.2. *Qualifying and quantifying two-mode entanglement*

5.2.1. Partial transposition and negativities. The PPT condition for separability, equation (58) has obviously a very simple form for two-mode Gaussian states. In terms of symplectic invariants, partial transposition corresponds to flipping the sign of $\mathrm{Det}\,\boldsymbol{\gamma}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix} \quad \xrightarrow{\varrho \to \varrho^{\mathsf{T}_i}} \quad \tilde{\boldsymbol{\sigma}} = \begin{pmatrix} \boldsymbol{\alpha} & \tilde{\boldsymbol{\gamma}} \\ \tilde{\boldsymbol{\gamma}}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix}, \tag{78}$$

with $\mathrm{Det}\,\tilde{\boldsymbol{\gamma}} = -\mathrm{Det}\,\boldsymbol{\gamma}$. For a standard-form CM, equation (72), this simply means $c_+ \to c_+, c_- \to c_-$. Accordingly, the seralian $\Delta = \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} + 2\,\mathrm{Det}\,\boldsymbol{\gamma}$, equation (32), is mapped, under partial transposition, into

$$\begin{aligned} \tilde{\Delta} &= \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} + 2\,\mathrm{Det}\,\tilde{\boldsymbol{\gamma}} = \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} - 2\,\mathrm{Det}\,\boldsymbol{\gamma} \\ &= \Delta - 4\,\mathrm{Det}\,\boldsymbol{\gamma} = -\Delta + 2/\mu_1^2 + 2/\mu_2^2. \end{aligned} \tag{79}$$

From equation (75), the symplectic eigenvalues of the partial transpose $\tilde{\boldsymbol{\sigma}}$ of a two-mode CM $\boldsymbol{\sigma}$ are promptly determined in terms of symplectic invariants [28],

$$2\tilde{\nu}_{\mp}^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - \frac{4}{\mu^2}}. \tag{80}$$

The PPT criterion is then reexpressed by the following inequality:

$$\tilde{\Delta} \leqslant 1 + 1/\mu^2, \tag{81}$$

equivalent to separability. The state $\boldsymbol{\sigma}$ is separable if and only if $\tilde{\nu}_- \geqslant 1$. Accordingly, the logarithmic negativity equation (68) is a decreasing function of $\tilde{\nu}_-$ alone,

$$E_{\mathcal{N}} = \max\{0, -\log \tilde{\nu}_-\}, \tag{82}$$

as for the biggest symplectic eigenvalue of the partial transpose one has $\tilde{\nu}_+ > 1$ for all two-mode Gaussian states [37, 104].

Note that from equations (72), (73), (79) and (81) the following necessary condition for two-mode entanglement follows [13]:

$$\boldsymbol{\sigma} \text{ entangled} \quad \Rightarrow \quad \mathrm{Det}\,\boldsymbol{\gamma} < 0. \tag{83}$$

5.2.2. Entanglement of formation for symmetric states. The optimal convex decomposition involved in the definition, equation (62), of the entanglement of formation [54] has been remarkably solved in the special instance of two-mode symmetric mixed Gaussian states (i.e. with $\mathrm{Det}\,\boldsymbol{\alpha} = \mathrm{Det}\,\boldsymbol{\beta}$ in equation (72)) and turns out to be Gaussian. Namely, the absolute minimum is realized within the set of pure two-mode Gaussian states [97], yielding

$$E_F = \max[0, h(\tilde{\nu}_-)], \tag{84}$$

with

$$h(x) = \frac{(1+x)^2}{4x} \log\left[\frac{(1+x)^2}{4x}\right] - \frac{(1-x)^2}{4x} \log\left[\frac{(1-x)^2}{4x}\right]. \tag{85}$$

Such a quantity is, again, a monotonically decreasing function of the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partial transpose $\tilde{\boldsymbol{\sigma}}$ of a two-mode symmetric Gaussian CM $\boldsymbol{\sigma}$, thus providing a quantification of the entanglement of symmetric states *equivalent* to the one provided by the negativities. Lower bounds on the entanglement of formation have been derived for nonsymmetric two-mode Gaussian states [107].

As a consequence of the equivalence between negativities and Gaussian measures of entanglement in symmetric two-mode Gaussian states, it is tempting to conjecture that there exists a unique quantification of entanglement for all two-mode Gaussian states, embodied by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM, and that the different measures simply provide trivial rescalings of the same unique quantification. In particular, the *ordering* induced on the set of entangled Gaussian state is uniquely defined for the subset of symmetric two-mode states and it is independent of the chosen measure of entanglement. However, in section 5.4 we will indeed show, within the general framework of Gaussian measures of entanglement (see section 4.5.2), that different families of entanglement monotones induce, in general, different orderings on the set of nonsymmetric Gaussian states, as demonstrated in [105].

5.2.3. Gaussian measures of entanglement: geometric framework. The problem of evaluating Gaussian measures of entanglement (Gaussian EMs) for a generic two-mode Gaussian state has been solved in [106]. However, the explicit result contains rather cumbersome expressions, involving the solutions of a fourth-order algebraic equation. As such, they were judged of no particular insight to be reported explicitly by the authors of [106].

We recall here the computation procedure [105]. For any two-mode Gaussian state with CM $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_{sf}$ in standard form, equation (72), a generic Gaussian EM G_E is given by the entanglement E of the least entangled pure state with CM $\boldsymbol{\sigma}^p \leqslant \boldsymbol{\sigma}$, see equation (71). Denoting by γ_q (respectively γ_p) the 2×2 submatrix obtained from $\boldsymbol{\sigma}$ by cancelling the even (resp. odd) rows and columns, we have

$$\gamma_q = \begin{pmatrix} a & c_+ \\ c_+ & b \end{pmatrix}, \qquad \gamma_p = \begin{pmatrix} a & c_- \\ c_- & b \end{pmatrix}. \tag{86}$$

All the covariances relative to the 'position' operators of the two modes are grouped in γ_q, and analogously for the 'momentum' operators in γ_p. The total CM can then be written as a direct sum $\boldsymbol{\sigma} = \gamma_q \oplus \gamma_p$. Similarly, the CM of a generic pure two-mode Gaussian state in block-diagonal form (it has been proven that the CM of the optimal pure state has to be with all diagonal 2×2 submatrices as well [106]) can be written as $\boldsymbol{\sigma}^p = \gamma_q^p \oplus \gamma_p^p$, where the global purity of the state imposes $\left(\gamma_p^p\right)^{-1} = \gamma_q^p \equiv \Gamma$. The pure states involved in the definition of the Gaussian EM must thus fulfil the condition

$$\gamma_p^{-1} \leqslant \Gamma \leqslant \gamma_q. \tag{87}$$

This problem admits an interesting geometric formulation [106]. Writing the matrix Γ in the basis constituted by the identity matrix and the three Pauli matrices,

$$\Gamma = \begin{pmatrix} x_0 + x_3 & x_1 \\ x_1 & x_0 - x_3 \end{pmatrix}, \tag{88}$$

the expansion coefficients (x_0, x_1, x_3) play the role of coordinates in a three-dimensional Minkowski space. In this picture, for example, the rightmost inequality in equation (87) is satisfied by matrices Γ lying on a cone, which is equivalent to the (backwards) light cone of C_q in the Minkowski space, and similarly for the leftmost inequality. Indeed, one can show that, for the optimal pure state $\boldsymbol{\sigma}^p_{\text{opt}}$ realizing the minimum in equation (71), the two inequalities in equation (87) have to be simultaneously saturated [106]. From a geometrical point of view, the optimal Γ has then to be found on the rim of the intersection of the forward and the backward cones of γ_p^{-1} and γ_q, respectively. This is an ellipse, and one is left with the task of minimizing the entanglement E of $\boldsymbol{\sigma}^p = \Gamma \oplus \Gamma^{-1}$ (see equation (70)) for Γ lying on this ellipse[6].

[6] The geometric picture describing the optimal two-mode state which enters in the determination of the Gaussian EMs is introduced in [106]. A more detailed discussion, including the explicit expression of the Lorentz boost needed to move into the plane of the ellipse, can be found in [108].

We recall that any pure two-mode Gaussian state $\boldsymbol{\sigma}^p$ is locally equivalent to a two-mode squeezed state with squeezing parameter r, described by the CM of equation (21). The following statements are then equivalent: (i) E is a monotonically increasing function of the entropy of entanglement; (ii) E is a monotonically increasing function of the single-mode determinant $m^2 \equiv \operatorname{Det}\boldsymbol{\alpha} \equiv \operatorname{Det}\boldsymbol{\beta}$ (see equation (72)); (iii) E is a monotonically decreasing function of the local purity $\mu_i \equiv \mu_1 \equiv \mu_2$ (see equation (46)); (iv) E is a monotonically decreasing function of the smallest symplectic eigenvalue $\tilde{\nu}_-^p$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}^p$; (v) E is a monotonically increasing function of the squeezing parameter r. This chain of equivalences is immediately proven by simply recalling that a pure state is completely specified by its single-mode marginals and that for a single-mode Gaussian state there is a unique symplectic invariant (the determinant), so that all conceivable entropic quantities are monotonically increasing functions of this invariant, as shown in section 3.3.2 [37]. In particular, statement (ii) allows us to directly minimize the single-mode determinant over the ellipse:

$$m^2 = 1 + \frac{x_1}{\operatorname{Det}\Gamma}, \tag{89}$$

with Γ given by equation (88).

To simplify the calculations, one can move to the plane of the ellipse with a Lorentz boost which preserves the relations between all the cones; one can then choose the transformation so that the ellipse degenerates into a circle (with fixed radius) and introduce polar coordinates on this circle. The calculation of the Gaussian EM for any two-mode Gaussian state is thus finally reduced to the minimization of m^2 from equation (89), at given standard-form covariances of $\boldsymbol{\sigma}$, as a function of the polar angle θ on the circle [108]. After some tedious but straightforward algebra, one finds [105]

$$\begin{aligned} m_\theta^2(a,b,c_+,c_-) = 1 &+ \left\{[c_+(ab-c_-^2) - c_- + \cos\theta\sqrt{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}]^2\right\} \\ &\times \left\{2(ab-c_-^2)(a^2+b^2+2c_+c_-) \right. \\ &- \frac{\cos\theta[2abc_-^3 + (a^2+b^2)c_+c_-^2 + ((1-2b^2)a^2+b^2)c_- - ab(a^2+b^2-2)c_+]}{\sqrt{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}} \\ &\left. + \sin\theta(a^2-b^2)\sqrt{1-\frac{[c_+(ab-c_-^2)+c_-]^2}{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}}\right\}^{-1}, \end{aligned} \tag{90}$$

where we have assumed $c_+ \geqslant |c_-|$ without loss of generality. Therefore, for any entangled state, $c_+ > 0$ and $c_- < 0$, see equation (83). The Gaussian EM (defined in terms of the function E on pure states, see equation (70)) of a generic two-mode Gaussian state coincides then with the entanglement E computed on the pure state with $m^2 = m_{\text{opt}}^2$, $m_{\text{opt}}^2 \equiv \min_\theta\left(m_\theta^2\right)$. Accordingly, the symplectic eigenvalue $\tilde{\nu}_-$ of the partial transpose of the corresponding optimal pure-state CM $\boldsymbol{\sigma}_{\text{opt}}^p$, realizing the infimum in equation (71), would read (see equation (80))

$$\tilde{\nu}_{-\text{opt}}^p \equiv \tilde{\nu}_-\left(\boldsymbol{\sigma}_{\text{opt}}^p\right) = m_{\text{opt}} - \sqrt{m_{\text{opt}}^2 - 1}. \tag{91}$$

As an example, for the Gaussian entanglement of formation [106] one has

$$G_{E_F}(\boldsymbol{\sigma}) = h\left[\tilde{\nu}_{-\text{opt}}^p\left(m_{\text{opt}}^2\right)\right], \tag{92}$$

with $h(x)$ defined by equation (85).

Finding the minimum of equation (90) analytically for a generic state is an involved task, but numerical optimizations may be quite accurate.

5.3. Extremal entanglement versus information

Here, we review the characterization of entanglement of two-mode Gaussian states with respect to its relationship with the degrees of information associated with the global state of the system and with the reduced states of each of the two subsystems.

As extensively discussed in the previous sections, the concepts of entanglement and information encoded in a quantum state are closely related. Specifically, for pure states bipartite entanglement is equivalent to the lack of information (mixedness) of the reduced state of each subsystem. For mixed states, each subsystem has its own level of impurity, and moreover the global state is itself characterized by a nonzero mixedness. Each of these properties can be interpreted as information on the preparation of the respective (marginal and global) states, as clarified in section 3.3.1. Therefore, by properly accessing these degrees of information it should be possible to deduce, at least to some extent, the status of the correlations between the subsystems.

Indeed, the original studies revealed that, at fixed global and marginal degrees of purity (or of generalized entropies), the negativities of arbitrary (mixed) two-mode Gaussian states are analytically constrained by rigorous upper and lower bounds [27, 37, 104]. This follows by reparametrizing, as already anticipated, the standard-form CM equation (72) in terms of the invariants $\mu_1, \mu_2, \mu\Delta$ and by observing that, at fixed purities, the negativities are monotonically decreasing function of Δ. Further constraints imposed on Δ by the uncertainty principle and by the existence condition of the radicals involved in the reparametrization,

$$\frac{2}{\mu} + \frac{(\mu_1 - \mu_2)^2}{\mu_1^2 \mu_2^2} \leqslant \Delta \leqslant \min\left\{\frac{(\mu_1 + \mu_2)^2}{\mu_1^2 \mu_2^2} - \frac{2}{\mu}, 1 + \frac{1}{\mu^2}\right\}, \tag{93}$$

immediately lead to the definition of *extremally*—maximally and minimally—entangled Gaussian states at fixed global and local purities. They are known, respectively, as 'GMEMS' (saturating the leftmost inequality in equation (93)), alias nonsymmetric thermal squeezed states, and 'GLEMS' (saturating the rightmost inequality in equation (93)), alias mixed states of partial minimum uncertainty [37, 104]. Nonsymmetric thermal squeezed states have also been proven to be maximally entangled Gaussian mixed states at fixed global purity and mean energy [109].

Summarizing, *the entanglement, quantified by the negativities, of two-mode (mixed) Gaussian states is strictly bound from above and from below by the amounts of global and marginal purities, with only one remaining degree of freedom related to the symplectic invariant* Δ.

The existence of GMEMS and GLEMS has two consequences. First, it allows for a classification of the properties of separability of all two-mode Gaussian states according to their degree of global and marginal purities, as summarized in table 4. Namely, from the separability properties of the extremally entangled states, necessary and/or sufficient conditions for entanglement—which constitute the strongest entropic criteria for separability [110] to date in the case of Gaussian states—are straightforwardly derived, which allow one to decide almost unambiguously if a given two-mode Gaussian state is entangled or not based on its degree of global and local purities. There is only a narrow region where, for given purities, both separable and entangled states can coexist, as pictorially shown in figure 1.

The second consequence is of a quantitative nature. The quantitative analysis of the maximal and the minimal entanglement allowed for a Gaussian state with given purities reveals that they are very close to each other, their difference narrowing exponentially with increasing entanglement. One can then define the *average logarithmic negativity* (mean value of the entanglements of the GMEMS and the GLEMS corresponding to a given triplet of purities) as a reliable estimator of bipartite entanglement in two-mode Gaussian states and its accurate

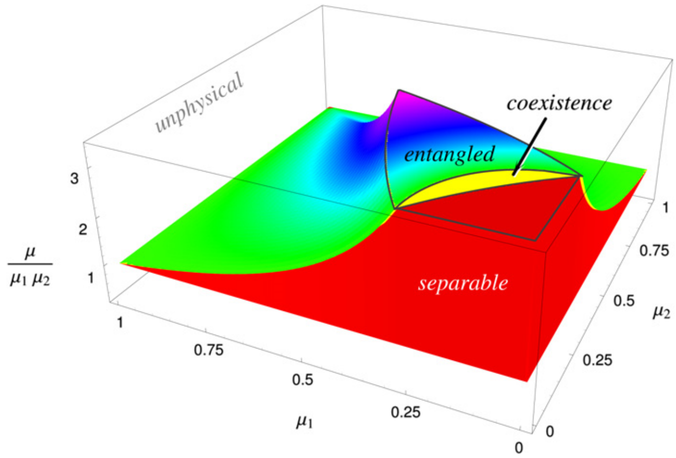


Figure 1. Summary of entanglement properties of two-mode (nonsymmetric) Gaussian states in the space of marginal purities $\mu_{1,2}$ and global purity μ (we plot the normalized ratio $\mu/\mu_1\mu_2$ to gain a better graphical clarity). In this space, separable states (red zone) and entangled states (green to magenta zone, according to the average entanglement) are well separated except for a narrow region of coexistence (depicted in yellow). The mathematical relations defining the boundaries between the three regions are collected in table 4. The three-dimensional envelope is cut at $z = 3.5$.

Table 4. Classification of two-mode Gaussian states and of their properties of separability according to their degrees of global purity μ and of marginal purities μ_1 and μ_2 [37, 104].

Degrees of purity	Entanglement properties
$\mu < \mu_1\mu_2$	Unphysical region
$\mu_1\mu_2 \leqslant \mu \leqslant \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2}$	*Separable* states
$\frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} < \mu \leqslant \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}}$	*Coexistence* region
$\frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}} < \mu \leqslant \frac{\mu_1\mu_2}{\mu_1\mu_2+\|\mu_1-\mu_2\|}$	*Entangled* states
$\mu > \frac{\mu_1\mu_2}{\mu_1\mu_2+\|\mu_1-\mu_2\|}$	Unphysical region

quantification by knowledge of the global and marginal purities alone [104]. The purities are nonlinear functionals of the state, but, assuming some prior knowledge about it (essentially, its Gaussian character), be measured through direct methods, in particular by means of single-photon detection schemes [111] (of which preliminary experimental verifications are available [112]) or by interferometric quantum-network architectures [113–115]. Very recently, a scheme to measure locally all symplectic invariants (and hence the entanglement) of two-mode Gaussian states has been proposed [116]. Note that no complete homodyne reconstruction [117] of the covariance matrix is needed in all these schemes.

The average estimate of the logarithmic negativity becomes indeed an *exact* quantification in the two important instances of GMEMS (nonsymmetric thermal squeezed states) and GLEMS (mixed states of partial minimum uncertainty), whose logarithmic negativity is completely determined as a function of the three purities alone [37, 104].

5.4. Ordering Gaussian states with measures of entanglement

The role of GMEMS and GLEMS in the characterization of entanglement of two-mode Gaussian states is not confined to the choice of the negativities. In fact, the Gaussian EMs for these two families of states can be obtained in closed form, as the optimization involved in equation (90) admits a rather simple analytical solutions in these two instances, as we have shown in [105]. One might therefore raise the question whether such extremally entangled states (with respect to the negativities) conserve their role once the entanglement is measured by the Gaussian EMs. The answer, somehow surprisingly, is a negative one: actually, there is a large region in the space of purities, where GMEMS are strictly less entangled than the corresponding GMEMS, when their entanglement is measured according to the Gaussian EMs.

The Gaussian EMs and the negativities are thus *not* equivalent for the quantification of entanglement in mixed, nonsymmetric two-mode Gaussian states [105]. The interpretation of this result needs to be discussed with some care. On the one hand, one could think that the ordering induced by the negativities is a natural one, due to the fact that such measures of entanglement are directly inspired by the necessary and sufficient PPT criterion for separability. Thus, one would expect that the ordering induced by the negativities should be preserved by any *bona fide* measure of entanglement, especially if one considers that the extremal states, GLEMS and GMEMS, have a clear physical interpretation. Therefore, as the Gaussian entanglement of formation is an upper bound to the true entanglement of formation, one could be tempted to take this result as an evidence that the latter is globally minimized on non-Gaussian decomposition, at least for GLEMS. However, this is only a qualitative/speculative argument: proving or disproving that the Gaussian entanglement of formation is the true one for any two-mode Gaussian state remains an open question [56].

On the other hand, one could take the simplest discrete-variable instance, constituted by a two-qubit system, as a test case for comparison. There, although for pure states the negativity coincides with the concurrence, an entanglement monotone equivalent to the entanglement of formation for all states of two qubits [118], the two measures cease to be equivalent for mixed states, and the orderings they induce on the set of entangled states can be different [119, 120]. This analogy seems to support the stand that, in the arena of mixed states, the definition of a unique measure of entanglement cannot really be pursued, due to the different operational meanings and physical processes (in the cases when it has been possible to identify them) that are associated with each definition: one could think, for instance, of the operational difference existing between the definitions of distillable entanglement and entanglement cost [54]. In other words, from this point of view, each inequivalent measure of entanglement introduced for mixed states should physically capture distinct aspects and forms of quantum correlations existing in these states. Gaussian EMs might then still be considered as proper measures of CV entanglement, adapted to a different context than negativities. This view seems particularly appropriate when constructing Gaussian EMs to investigate entanglement sharing in multipartite Gaussian states [121, 122], as discussed in section 7.

The inequivalence between the two families of CV entanglement measures is somehow tempered. Namely, we have rigorously proven that, at fixed negativities, the Gaussian measures of entanglement are bounded from below (the states which saturate this bound are simply symmetric two-mode states); moreover, we provided some strong evidence suggesting that they are as well bounded from above [105]. A direct comparison between the logarithmic negativity and the Gaussian entanglement of formation for all two-mode Gaussian states is shown in figure 2.

6. Multimode bipartite entanglement: localization and scaling

We shall now discuss the properties of entanglement in multimode Gaussian states endowed with particular symmetry constraints under permutations of the modes [45, 46]. The usefulness of these states arise in contexts like quantum error correction [123], where some redundancy is

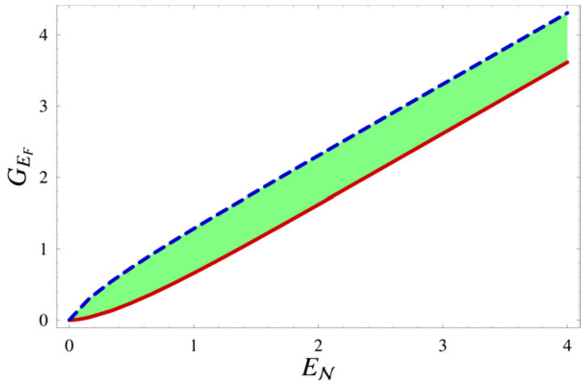


Figure 2. Comparison between the Gaussian entanglement of formation G_{E_F} and the logarithmic negativity $E_{\mathcal{N}}$ for two-mode Gaussian states. Symmetric states accommodate on the lower boundary (solid line). States of maximal negativities at fixed (infinite) average local mixedness lie on the dashed line. All GMEMS and GLEMS lie below the dashed line. The latter is conjectured, with strong numerical support, to be the upper boundary for the Gaussian entanglement of formation of all two-mode Gaussian states, at fixed negativity. Cf [105] for the complete discussion and the detailed mathematical proofs.

required for a fault-tolerant encoding of information. Bisymmetric and, as a special case, fully symmetric Gaussian states have been introduced in section 3.4.2. The study of the symplectic spectra of $(M + N)$-mode Gaussian states reveals that, with respect to the bipartition across which they exhibit the local permutation invariance (any bipartition is valid for fully symmetric states), local symplectic diagonalizations of the M-mode and the N-mode blocks result in a complete reduction of the multimode state to an equivalent two-mode state, tensored with $M + N - 2$ uncorrelated thermal single-mode states. The equivalent two-mode state possesses all the information of the original bisymmetric multimode state for what concerns entropy and entanglement. As a consequence, the validity of the PPT criterion as a necessary and sufficient condition for separability has been extended to bisymmetric Gaussian states in section 4.4.1.

6.1. Unitarily localizable entanglement of Gaussian states

Here, equipped with the tools introduced for the analysis of two-mode entanglement in Gaussian states, we discuss multimode entanglement in symmetric and bisymmetric Gaussian states. In particular, we will investigate how the block entanglement scales with the number of modes at fixed squeezing. The form of the scaling hints at the presence of genuine multipartite entanglement that progressively arises among all the modes as their total number increases.

The central observation of the present section is contained in the following result [45, 46], straightforwardly deducible from the discussions in sections 3.4.2 and 4.4.1. *The bipartite entanglement of bisymmetric $(M + N)$-mode Gaussian states under $M \times N$ partitions is 'unitarily localizable', namely, through local unitary (reversible) operations, it can be completely concentrated onto a single pair of modes, each of them belonging respectively to the M-mode and to the N-mode blocks.* Hence, the multimode block entanglement (i.e. the bipartite entanglement between two blocks of modes) of bisymmetric (generally mixed) Gaussian states can be determined as an equivalent two-mode entanglement. The entanglement will be quantified by the logarithmic negativity in the general instance because the PPT criterion holds, but we will also show some explicit nontrivial cases in which the entanglement of formation, equation (62), between M-mode and N-mode parties can be exactly computed.

We remark that the notion of 'unitarily localizable entanglement' is different from that introduced by Verstraete, Popp and Cirac for spin systems [124]. There, it was defined as the maximal entanglement that can be concentrated on two chosen spins through local *measurements* on all the other spins. Here, the local operations that concentrate all the multimode entanglement on two modes are *unitary* and involve the two chosen modes as well, as parts of the respective blocks. The localizable entanglement in the sense of [124] can be computed as well for (mixed) symmetric Gaussian states of an arbitrary number of modes, and is in direct quantitative connection with the optimal fidelity of multiparty teleportation networks [125].

It is important to observe that the unitarily localizable entanglement (when computable) is always stronger than the localizable entanglement in the sense of [124]. In fact, if we consider a generic bisymmetric multimode state of an $M \times N$ bipartition, with each of the two target modes owned respectively by one of the two parties (blocks), then the ensemble of optimal local measurements on the remaining ('assisting') $M + N - 2$ modes belongs to the set of LOCC with respect to the considered bipartition. By definition the entanglement cannot increase under LOCC, which implies that the localized entanglement (in the sense of [124]) is always less or equal than the original $M \times N$ block entanglement. In contrast, *all* of the same $M \times N$ original bipartite entanglement can be unitarily localized onto the two target modes.

This is a key point, as such local unitary transformations are *reversible* by definition. Therefore, by only using passive and active linear optics elements such as beam splitters, phase shifters and squeezers, one can in principle implement a reversible machine (*entanglement switch*) that, from mixed, bisymmetric multimode states with strong quantum correlations between all the modes (and consequently between the M-mode and the N-mode partial blocks) but weak couplewise entanglement, is able to extract a highly pure, highly entangled two-mode state (with no entanglement loss, as all the $M \times N$ entanglement can be localized). If needed, the same machine would be able, starting from a two-mode squeezed state and a collection of uncorrelated thermal or squeezed states, to distribute the two-mode entanglement between all modes, converting the two-mode into multimode, multipartite quantum correlations, again with no loss of entanglement. The bipartite or multipartite entanglement can then be used on demand, the first for instance in a CV quantum teleportation protocol [126], the latter to secure quantum key distribution or to perform multimode entanglement swapping [127]. We remark, once more, that such an entanglement switch is endowed with maximum (100%) efficiency, as no entanglement is lost in the conversions.

6.2. *Quantification and scaling of entanglement in fully symmetric states*

Here, we will review the explicit evaluation of the bipartite block entanglement for some instances of multimode Gaussian states. We will discuss its scaling behaviour as a function of the number of modes and explore in some detail the localizability of the multimode entanglement. We focus our attention on fully symmetric L-mode Gaussian states (the number of modes is denoted by L in general to avoid confusion), endowed with complete permutation invariance under mode exchange, and described by a $2L \times 2L$ CM $\boldsymbol{\sigma}_{\beta^L}$ given by equation (51). These states are trivially bisymmetric under any bipartition of the modes, so that their block entanglement is always localizable by means of local symplectic operations. Let us recall that concerning the covariances in normal forms of fully symmetric states (see section 3.4.2) *pure* L-mode states are characterized by

$$z_1 = \frac{(L-2)(b^2-1) + \sqrt{(b^2-1)[L((b^2-1)L+4)-4]}}{2b(L-1)},$$

$$z_2 = \frac{(L-2)(b^2-1) - \sqrt{(b^2-1)[L((b^2-1)L+4)-4]}}{2b(L-1)}. \tag{94}$$

Pure, fully symmetric Gaussian states are generated as the outputs of the application of a sequence of $L-1$ beam splitters to L single-mode squeezed inputs [90, 128]. The CM $\boldsymbol{\sigma}^p_{\beta^L}$ of this class of pure states, for a given number of modes, depends only on the parameter $b \equiv 1/\mu_\beta \geqslant 1$, which is an increasing function of the single-mode squeezing needed to prepare the state. Correlations between the modes are induced according to the above expression for the covariances z_i. Their multipartite entanglement sharing has been studied in [121] (see the next section), and their use in teleportation networks has been investigated in [125, 128].

In general, one can compute the entanglement between a block of K modes and the remaining $L-K$ modes for pure states (in this case the block entanglement is simply equivalent to the von Neumann entropy of each of the reduced blocks) and for mixed, fully symmetric states under any bipartition of the modes.

6.2.1. $1\times N$ entanglement. Based on [46], we begin by assigning a single mode to subsystem $\mathcal{S}_A$, and an arbitrary number N of modes to subsystem $\mathcal{S}_B$, forming a CV system globally prepared in a fully symmetric $(1+N)$-mode Gaussian state of modes.

We consider pure, fully symmetric states with CM $\boldsymbol{\sigma}^p_{\beta^{1+N}}$, obtained by inserting equation (94) with $L \equiv (1+N)$. Exploiting our previous analysis, we can compute the entanglement between a single mode with reduced CM $\boldsymbol{\sigma}^\beta$ and any K-mode partition of the remaining modes ($1 \leqslant K \leqslant N$), by determining the equivalent two-mode CM $\boldsymbol{\sigma}^{\beta|\beta^K}_{\rm eq}$. We remark that, for every K, the $1\times K$ entanglement is always equivalent to a 1×1 entanglement, so that the quantum correlations between the different partitions of $\boldsymbol{\sigma}$ can be directly compared to each other: it is thus possible to establish a multimode entanglement hierarchy without any problem of ordering.

The $1\times K$ entanglement quantified by the logarithmic negativity $E_{\mathcal{N}}^{\beta|\beta^K}$ is determined by the smallest symplectic eigenvalue $\tilde{\nu}_-^{(K,N)}$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}^{\beta|\beta^K}_{\rm eq}$. For any nonzero squeezing (i.e., $b>1$) one has that $\tilde{\nu}_-^{(K,N)} < 1$, meaning that the state exhibits genuine multipartite entanglement: each mode is entangled with any other K-mode block, as first remarked in [90]. Further, the genuine multipartite nature of the entanglement can be precisely quantified by observing that

$$E_{\mathcal{N}}^{\beta|\beta^K} \geqslant E_{\mathcal{N}}^{\beta|\beta^{K-1}},$$

as shown in figure 3.

The 1×1 entanglement between two modes is weaker than the 1×2 one between a mode and other two modes, which is in turn weaker than the $1\times K$ one, and so on with increasing K in this typical cascade structure. From an operational point of view, a signature of *genuine multipartite entanglement* is revealed by the fact that performing, e.g., a local measurement on a single mode will affect *all* the other N modes. This means that the quantum correlations contained in the state with CM $\boldsymbol{\sigma}^p_{\beta^{1+N}}$ can be fully recovered only when considering the $1\times N$ partition.

In particular, the pure-state $1\times N$ logarithmic negativity is, as expected, independent of N, being a simple monotonic function of the entropy of entanglement E_V, equation (53) (defined as the von Neumann entropy of the reduced single-mode state with CM $\boldsymbol{\sigma}_\beta$). It is worth noting that, in the limit of infinite squeezing ($b\to\infty$), only the $1\times N$ entanglement diverges while all the other $1\times K$ quantum correlations remain finite (see figure 3). Namely,

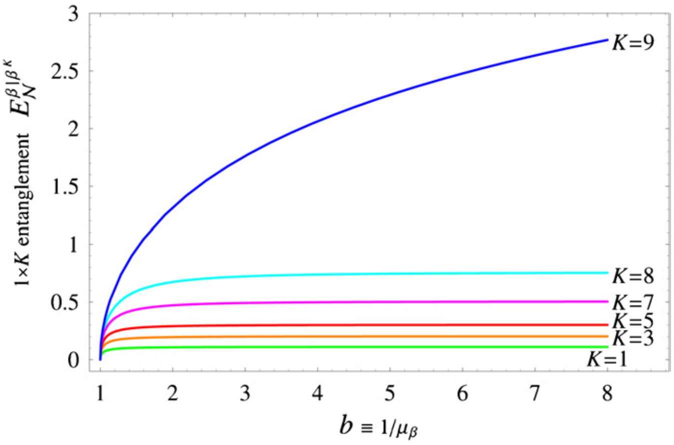


Figure 3. Entanglement hierarchy for $(1+N)$-mode fully symmetric pure Gaussian states ($N = 9$).

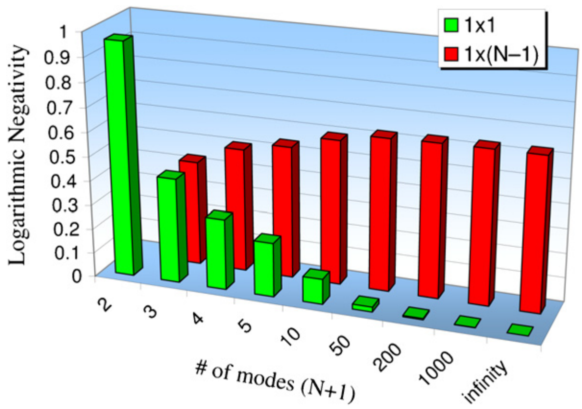


Figure 4. Scaling as a function of N of the 1×1 entanglement (green bars) and of the $1 \times (N-1)$ entanglement (red bars) for a $(1+N)$-mode pure fully symmetric Gaussian state, at fixed squeezing ($b = 1.5$).

$$E_{\mathcal{N}}^{\beta|\beta^K}\left(\boldsymbol{\sigma}^p_{\beta^{1+N}}\right) \stackrel{b\to\infty}{\longrightarrow} -\frac{1}{2}\log\left[\frac{1-4K}{N(K+1)-K(K-3)}\right], \tag{95}$$

which cannot exceed $\log\sqrt{5} \simeq 0.8$ for any N and for any $K < N$.

At fixed squeezing (i.e. fixed local properties, $b \equiv 1/\mu_\beta$), the *scaling* with N of the $1 \times (N-1)$ entanglement compared to the 1×1 entanglement is shown in figure 4 (we recall that the $1 \times N$ entanglement is independent of N). Note how, with increasing number of modes, the multimode entanglement increases to the detriment of the two-mode one. The latter is indeed being *distributed* among all the modes: this feature will be properly quantified within the framework of CV entanglement sharing in the next section [121].

We remark that such a scaling feature occurs both in fully symmetric and bisymmetric states (think, for instance, to a single-mode squeezed state coupled with an N-mode symmetric thermal squeezed state), pure or mixed. The simplest example of a mixed state in which our analysis reveals the presence of genuine multipartite entanglement is obtained from $\boldsymbol{\sigma}^p_{\beta^{1+N}}$ by

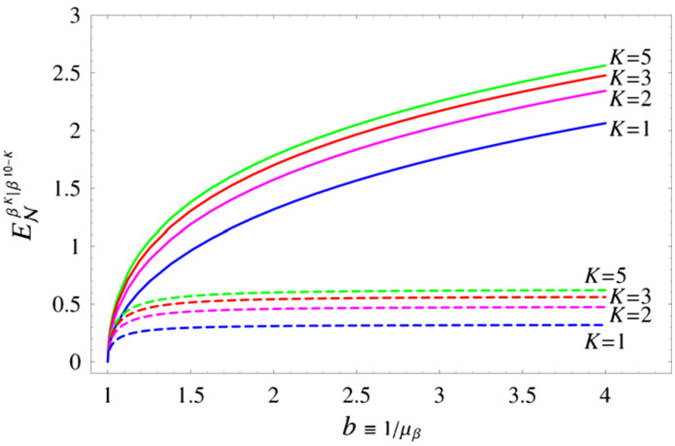


Figure 5. Hierarchy of block entanglements of fully symmetric $2N$-mode Gaussian states of $K \times (2N - K)$ bipartitions ($2N = 10$) as a function of the single-mode squeezing b. The block entanglements are depicted both for pure states (solid lines) and for mixed states obtained from fully symmetric $(2N + 4)$-mode pure Gaussian states by tracing out four modes (dashed lines).

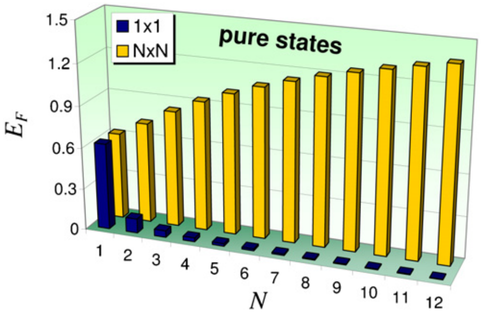
tracing out some of the modes. Figure 4 can then also be seen as a demonstration of the scaling in such an N-mode mixed state, where the $1 \times (N - 1)$ entanglement is the strongest one. Thus, with increasing N, the global mixedness can limit but not destroy the distribution of entanglement in multiparty form among all the modes.

6.2.2. $M \times N$ entanglement. Based on the results of [45], one can consider a generic $2N$-mode fully symmetric mixed state with CM $\boldsymbol{\sigma}_{\beta^{2N}}^{p\backslash Q}$, see equation (51), obtained from a pure fully symmetric $(2N + Q)$-mode state by tracing out Q modes.

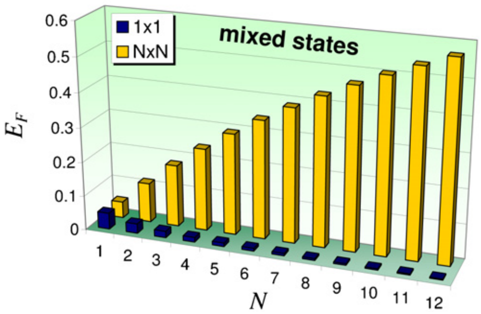
For any Q, for any dimension K of the block ($K \leqslant N$) and for any nonzero squeezing (i.e. for $b > 1$) one has that $\tilde{\nu}_K < 1$, meaning that the state exhibits genuine multipartite entanglement, generalizing the $1 \times N$ case described before: each K-mode party is entangled with the remaining $(2N-K)$-mode block. The genuine multipartite nature of the entanglement can be determined by observing that, again, $E_{\mathcal{N}}^{\beta^K|\beta^{2N-K}}$ is an increasing function of the integer $K \leqslant N$, as shown in figure 5. The multimode entanglement of mixed states also remains finite in the limit of infinite squeezing, while the multimode entanglement of pure states diverges with respect to any bipartition, as shown in figure 5.

In fully symmetric Gaussian states, the block entanglement is unitarily localizable with respect to any $K \times (2N - K)$ bipartition. Since in this instance *all* the entanglement can be concentrated on a single pair of modes, after the partition has been decided, no strategy could grant a better yield than the local symplectic operations bringing the reduced CMs in Williamson form (because of the monotonicity of the entanglement under general LOCC). However, the amount of block entanglement, which is the amount of concentrated two-mode entanglement after unitary localization has taken place, actually depends on the choice of a particular $K \times (2N - K)$ bipartition, giving rise to a hierarchy of localizable entanglements.

Let us imagine that for a given Gaussian multimode state (say, for simplicity, a fully symmetric state) its entanglement is meant to serve as a resource for a given protocol. Let us next suppose that the protocol is optimally implemented if the entanglement is concentrated between only two modes of the global systems, as it is the case, e.g., in a CV teleportation protocol between two single-mode parties [126]. Which choice of the bipartition between



(*a*)



(*b*)

Figure 6. Scaling, with half the number of modes, of the entanglement of formation in two families of fully symmetric $2N$-mode Gaussian states. Plot (*a*) depicts pure states, while mixed states (*b*) are obtained from $(2N + 4)$-mode pure states by tracing out four modes. For each class of states, two sets of data are plotted, one referring to $N \times N$ entanglement (yellow bars) and the other to 1×1 entanglement (blue bars). Note how the $N \times N$ entanglement, equal to the optimal localizable entanglement (OLE) and estimator of genuine multipartite quantum correlations among all the $2N$ modes, increases at the detriment of the bipartite 1×1 entanglement between any pair of modes. The single-mode squeezing parameter is fixed at $b = 1.5$.

the modes allows for the best entanglement concentration by a succession of local unitary operations? In this framework, it turns out that assigning $K = 1$ mode at one party and all the remaining modes to the other, as discussed in section 6.2.1, constitutes the *worst* localization strategy [45]. Conversely, for an even number of modes the best option for localization is an equal $K = N$ splitting of the $2N$ modes between the two parties. The logarithmic negativity $E_{\mathcal{N}}^{\beta^N|\beta^N}$, concentrated into two modes by local operations, represents the optimal localizable entanglement (OLE) of the state $\boldsymbol{\sigma}_{\beta^{2N}}$, where 'optimal' refers to the choice of the bipartition. Clearly, the OLE of a state with $2N + 1$ modes is given by $E_{\mathcal{N}}^{\beta^{N+1}|\beta^N}$. These results may be applied to arbitrary, pure or mixed, fully symmetric Gaussian states.

We now turn to the study of the scaling behaviour of the OLE of $2N$-mode Gaussian states when the number of modes is increased, to understand how the number of local cooperating parties can improve the maximal entanglement that can be shared between two parties. For generic (mixed) fully symmetric $2N$-mode states of $N \times N$ bipartitions, the OLE can also be quantified by the entanglement of formation E_F, equation (84), as the equivalent two-mode state is symmetric [45]. It is then useful to compare, as a function of N, the 1×1 entanglement of formation between a pair of modes (all pairs are equivalent due to the global symmetry of the state) before the localization, and the $N \times N$ entanglement of formation, which is equal to the optimal entanglement concentrated in a specific pair of modes after performing the local unitary operations. The results of this study are shown in figure 6. The two quantities are plotted at fixed squeezing b as a function of N both for a pure $2N$-mode state with CM $\boldsymbol{\sigma}^{p}_{\beta^{2N}}$ and a mixed $2n$-mode state with CM $\boldsymbol{\sigma}^{p\backslash 4}_{\beta^{2N}}$. As the number of modes increases, any pair of single modes becomes steadily less entangled, but the total multimode entanglement of the state grows and, as a consequence, the OLE increases with N. In the limit $N \to \infty$, the $N \times N$ entanglement diverges while the 1×1 one vanishes. This exactly holds both for pure *and* mixed states, although the global degree of mixedness produces the typical behaviour that tends to reduce the total entanglement of the state.

6.3. Towards multipartite entanglement

We have shown that bisymmetric multimode Gaussian states (pure or mixed) can be reduced, by local symplectic operations, to the tensor product of a correlated two-mode Gaussian state and of uncorrelated thermal states (the latter being obviously irrelevant as far as the correlation properties of the multimode Gaussian state are concerned). As a consequence, *all* the entanglement of bisymmetric multimode Gaussian states of arbitrary $M \times N$ bipartitions is *unitarily localizable* in a single (arbitrary) pair of modes shared by the two parties. Such a useful reduction to two-mode Gaussian states is somehow similar to the one holding for states with fully degenerate symplectic spectra [40, 41], encompassing the relevant instance of pure states, for which all the symplectic eigenvalues are equal to 1. The present result allows us to extend the PPT criterion as a necessary and sufficient condition for separability for all bisymmetric multimode Gaussian states of arbitrary $M \times N$ bipartitions (as shown in section 4.4.1) and to quantify their entanglement [45, 46].

Note that, in the general bisymmetric instance addressed in this section, the possibility of performing a two-mode reduction is crucially partition dependent. However, as we have explicitly shown, in the case of fully symmetric states all the possible bipartitions can be analysed and compared, yielding remarkable insight into the structure of the multimode block entanglement of Gaussian states. This finally leads to the determination of the maximum or optimal localizable entanglement that can be concentrated on a single pair of modes.

The multipartite entanglement in the considered class of multimode Gaussian states can be produced, detected [90, 128] and, by virtue of the present analysis, reversibly localized by all-optical means. Moreover, the multipartite entanglement allows for a reliable (i.e. with fidelity $\mathcal{F} > \mathcal{F}_{\rm cl}$, where $\mathcal{F}_{\rm cl} = 1/2$ is the classical threshold [129, 130]) quantum teleportation between any two parties with the assistance of the remaining others [128]. The connection between entanglement in the symmetric Gaussian resource states and optimal teleportation-network fidelity has been clarified in [125], as briefly recalled in section 9.2.

The present section is intended as a bridge between the two main parts of this review, the one dealing with bipartite entanglement and the one dealing with multipartite entanglement. We have characterized entanglement in multimode Gaussian states by reducing it to a two-mode problem. By comparing the equivalent two-mode entanglements in the different bipartitions one is unambiguously able to detect the presence of genuine multipartite entanglement. We now analyse in more detail the sharing phenomenon responsible for the distribution of entanglement from a bipartite, two-mode form, to a genuine multipartite manifestation in N-mode Gaussian states, both with and without symmetry constraints.

7. Distributed entanglement and monogamy inequality for all Gaussian states

We will now review some recent work on multipartite entanglement sharing in CV systems that has lead to the definition of a mathematically and physically *bona fide* measure of genuine tripartite entanglement for arbitrary three-mode Gaussian states [121, 131], the proof of the monogamy inequality on distributed entanglement for all Gaussian states [122] and the demonstration of the *promiscuous* sharing structure of multipartite entanglement in Gaussian states [121], which can be unlimited in states of more than three modes [132].

We begin by introducing some new entanglement monotones, the *contangle*, the *Gaussian contangle* and the *Gaussian tangle* [121, 122], apt to quantify distributed Gaussian entanglement, thus generalizing to the CV setting the *tangle* [133] defined for systems of qubits.

Motivated by the analysis of the block entanglement hierarchy and its scaling structure in fully symmetric Gaussian states (see the previous section) we will review the central result that CV entanglement, once properly quantified, is monogamous for *all* Gaussian states, in the sense that it obeys a proper generalization to CV systems of the Coffman–Kundu–Wootters monogamy inequality [122]. In the next section, we will review the results concerning the simplest instance of tripartite CV entangled states, the three-mode Gaussian states [131]. For these states, thanks to the monogamy inequality, it is possible to construct a measure of genuine tripartite entanglement, the *residual contangle*, that has been shown to be a full entanglement monotone under Gaussian LOCC [121]. Equipped with such a powerful tool to quantify tripartite entanglement, we will proceed to review the entanglement sharing structure in three-mode Gaussian states and the related, peculiar property of *promiscuity* for CV entanglement. This property essentially consists in the fact that bipartite and multipartite entanglement in multimode Gaussian states can be enhanced and simultaneously maximized without violating the monogamy inequality on entanglement sharing and can even grow unboundedly in Gaussian states of more than three modes [132].

7.1. The need for a new continuous-variable entanglement monotone

Our primary aim, as in [121], is to analyse the distribution of entanglement between different (partitions of) modes in Gaussian states of CV systems.

In, [133] Coffman, Kundu and Wootters (CKW) proved, for system of three qubits, and conjectured, for N qubits (this conjecture has now been proven by Osborne and Verstraete [134]), that the bipartite entanglement E (properly quantified) between, say, qubit A and the remaining two-qubits partition (BC) is never smaller than the sum of the $A|B$ and $A|C$ bipartite entanglements in the reduced states:

$$E^{A|(BC)} \geqslant E^{A|B} + E^{A|C}. \tag{96}$$

This statement quantifies the so-called *monogamy* of quantum entanglement [135], in opposition to the classical correlations, which are not constrained and can be freely shared [136].

One would expect a similar inequality to hold for three-mode Gaussian states, namely

$$E^{i|(jk)} - E^{i|j} - E^{i|k} \geqslant 0, \tag{97}$$

where E is a proper measure of bipartite CV entanglement and the indices $\{i, j, k\}$ label the three modes. However, the demonstration of such a property is plagued by subtle difficulties.

Let us consider the simplest conceivable instance of a pure three-mode Gaussian state completely invariant under mode permutations. These pure Gaussian states are named fully symmetric (see section 3.4.2), and their standard-form CM (obtained by inserting equation (94) with $L = 3$ into equation (51)) is only parametrized by the local mixedness $b = (1/\mu_\beta) \geqslant 1$, an increasing function of the single-mode squeezing r_{loc}, with $b \to 1^+$ when $r_{\text{loc}} \to 0^+$. For these states, it is not difficult to show that the inequality (97) can be violated for small values of the local squeezing factor, using either the logarithmic negativity $E_\mathcal{N}$ or the entanglement of formation E_F (which is computable in this case via equation (84), because the two-mode reduced mixed states of a pure symmetric three-mode Gaussian states are again symmetric) to quantify the bipartite entanglement. This fact implies that none of these two measures is the proper candidate for approaching the task of quantifying entanglement sharing in CV systems. This situation is reminiscent of the case of qubit systems, for which the CKW inequality holds using the tangle τ, defined as the square of the concurrence [118], but can fail if one chooses equivalent measures of bipartite entanglement such as the concurrence itself or the entanglement of formation [133].

It is then necessary to define a proper measure of CV entanglement that specifically quantifies entanglement sharing according to a monogamy inequality of the form (97) [121]. A first important hint towards this goal comes by observing that, when dealing with $1 \times N$ partitions of fully symmetric multimode pure Gaussian states together with their 1×1 reduced partitions, the desired measure should be a monotonically decreasing function f of the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the corresponding partially transposed CM $\tilde{\boldsymbol{\sigma}}$. This requirement stems from the fact that $\tilde{\nu}_-$ is the only eigenvalue that can be smaller than 1, as shown in section 4.4.1, violating the PPT criterion with respect to the selected bipartition. Moreover, for a pure symmetric three-mode Gaussian state, it is necessary to require that the bipartite entanglements $E^{i|(jk)}$ and $E^{i|j} = E^{i|k}$ be respectively functions $f(\tilde{n}_-^{i|(jk)})$ and $f(\tilde{\nu}_-^{i|j})$ of the associated smallest symplectic eigenvalues $\tilde{\nu}_-^{i|(jk)}$ and $\tilde{\nu}_-^{i|j}$, in such a way that they become infinitesimal of the same order in the limit of vanishing local squeezing, together with their first derivatives:

$$\frac{f(\tilde{\nu}_-^{i|(jk)}}{2f(\tilde{\nu}_-^{i|j}} \simeq \frac{f'(\tilde{\nu}_-^{i|(jk)}}{2f'(\tilde{\nu}_-^{i|j}} \to 1 \qquad \text{for} \quad b \to 1^+, \tag{98}$$

where the prime denotes differentiation with respect to the single-mode mixedness b. The violation of the sharing inequality (97) exhibited by the logarithmic negativity can be in fact traced back to the divergence of its first derivative in the limit of vanishing squeezing. The above condition formalizes the physical requirement that in a *symmetric* state the quantum correlations should grow smoothly and be distributed uniformly among all the three modes. One can then see that the unknown function f exhibiting the desired property is simply the squared logarithmic negativity[7]

$$f(\tilde{\nu}_-) = [-\log \tilde{\nu}_-]^2. \tag{99}$$

We remind again that for (fully symmetric) $(1 + N)$-mode pure Gaussian states, the partially transposed CM with respect to any $1 \times N$ bipartition, or with respect to any reduced 1×1 bipartition, has only one symplectic eigenvalue that can drop below 1; hence the simple form of the logarithmic negativity (and, equivalently, of its square) in equation (99).

7.2. *Squared negativities as continuous-variable tangles*

Equipped with this finding, one can give a formal definition of a bipartite entanglement monotone E_τ that, as we will soon show, can be regarded as a CV analogue of the tangle. Note that the context here is completely general and we are not assuming that we are dealing

[7] Note that an infinite number of functions satisfying equation (98) can be obtained by expanding $f(\tilde{\nu}_-)$ around $\tilde{\nu}_- = 1$ at any even order. However, they are all monotonic convex functions of f. If the inequality (97) holds for f, it will hold as well for any monotonically increasing, convex function of f, such as the logarithmic negativity raised to any even power $k \geqslant 2$, but not for $k = 1$. We will exploit this 'gauge freedom' in the following to define an equivalent entanglement monotone in terms of squared negativity [122].

with Gaussian states only. For a generic pure state $|\psi\rangle$ of a $(1+N)$-mode CV system, we define the square of the logarithmic negativity (the latter defined by equation (66)):

$$E_\tau(\psi) \equiv \log^2 \|\tilde{\varrho}\|_1, \qquad \varrho = |\psi\rangle\langle\psi|. \tag{100}$$

This is a proper measure of bipartite entanglement, being a convex, increasing function of the logarithmic negativity $E_\mathcal{N}$, which is equivalent to the entropy of entanglement equation (53) for arbitrary pure states in any dimension. Definition (100) is naturally extended to generic mixed states ρ of $(N+1)$-mode CV systems through the convex-roof formalism. Namely, we can introduce the quantity

$$E_\tau(\rho) \equiv \inf_{\{p_i,\psi_i\}} \sum_i p_i E_\tau(\psi_i), \tag{101}$$

where the infimum is taken over all convex decompositions of ρ in terms of pure states $\{|\psi_i\rangle\}$, and if the index i is continuous the sum in equation (101) is replaced by an integral and the probabilities $\{p_i\}$ by a probability distribution $\pi(\psi)$. Let us now recall that, for two qubits, the tangle can be defined as the convex roof of the squared negativity [137] (the latter being equal to the concurrence [118] for pure two-qubit states [120, 137]). Here, equation (101) states that the convex roof of the squared logarithmic negativity properly defines the continuous-variable tangle, or, in short, the *contangle* $E_\tau(\rho)$, in which the logarithm takes into account for the infinite dimensionality of the underlying Hilbert space.

On the other hand, by recalling the equivalence of negativity and concurrence for pure states of qubits, the *tangle* itself can be defined for CV systems as the convex-roof extension of the squared negativity. Let us recall that the negativity $\mathcal{N}$, equation (63), of a quantum state ϱ is a convex function of the logarithmic negativity $E_\mathcal{N}$, equation (66),

$$\mathcal{N}(\varrho) = \frac{\exp[E_\mathcal{N}(\varrho) - 1]}{2}. \tag{102}$$

7.2.1. Gaussian contangle and Gaussian tangle. From now on, we will restrict our attention to Gaussian states.

Gaussian contangle. For any pure multimode Gaussian state $|\psi\rangle$, with CM $\boldsymbol{\sigma}^p$, of $N+1$ modes assigned in a generic $1 \times N$ bipartition, explicit evaluation gives immediately that $E_\tau(\psi) \equiv E_\tau(\boldsymbol{\sigma}^p)$ takes the form

$$E_\tau(\boldsymbol{\sigma}^p) = \log^2\left(1/\mu_1 - \sqrt{1/\mu_1^2 - 1}\right), \tag{103}$$

where $\mu_1 = 1/\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}_1}$ is the local purity of the reduced state of mode 1 with CM $\boldsymbol{\sigma}_1$.

For any multimode, mixed Gaussian states with CM $\boldsymbol{\sigma}$, we will then denote the contangle by $E_\tau(\boldsymbol{\sigma})$, in analogy with the notation used for the contangle $E_\tau(\boldsymbol{\sigma}^p)$ of pure Gaussian states in equation (103). Any multimode mixed Gaussian state with CM $\boldsymbol{\sigma}$ admits at least one decomposition in terms of pure Gaussian states $\boldsymbol{\sigma}^p$ only. The infimum of the average contangle, taken over all pure Gaussian state decompositions, then defines the *Gaussian contangle* G_τ:

$$G_\tau(\boldsymbol{\sigma}) \equiv \inf_{\{\pi(\mathrm{d}\boldsymbol{\sigma}^p),\boldsymbol{\sigma}^p\}} \int \pi(\mathrm{d}\boldsymbol{\sigma}^p) E_\tau(\boldsymbol{\sigma}^p). \tag{104}$$

It follows from the convex-roof construction that the Gaussian contangle $G_\tau(\boldsymbol{\sigma})$ is an upper bound to the true contangle $E_\tau(\boldsymbol{\sigma})$ (as the latter can be in principle minimized over a non-Gaussian decomposition),

$$E_\tau(\boldsymbol{\sigma}) \leqslant G_\tau(\boldsymbol{\sigma}). \tag{105}$$

It can be shown that $G_\tau(\boldsymbol{\sigma})$ is a bipartite entanglement monotone under Gaussian LOCC: in fact, the Gaussian contangle belongs to the general family of Gaussian entanglement measures, whose properties as studied in [105] have been presented in section 4.5.2. Therefore, for Gaussian states, the Gaussian contangle, similarly to the Gaussian entanglement of formation [106], takes the simple form

$$G_\tau(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}^p \leqslant \boldsymbol{\sigma}} E_\tau(\boldsymbol{\sigma}^p), \tag{106}$$

where the infimum runs over all pure Gaussian states with CM $\boldsymbol{\sigma}^p \leqslant \boldsymbol{\sigma}$. Let us remark that if $\boldsymbol{\sigma}$ denotes a mixed symmetric two-mode Gaussian state then the Gaussian decomposition is the optimal one [97] (see section 5.2.2), and the optimal pure-state CM $\boldsymbol{\sigma}^p$ minimizing $G_\tau(\boldsymbol{\sigma})$ is characterized by having $\tilde{\nu}_-(\tilde{\boldsymbol{\sigma}}^p) = \tilde{\nu}_-(\tilde{\boldsymbol{\sigma}})$ [106]. The fact that the smallest symplectic eigenvalue is the same for both partially transposed CMs entails that

$$E_\tau(\boldsymbol{\sigma}) = G_\tau(\boldsymbol{\sigma}) = [\max\{0, -\log \tilde{\nu}_-(\boldsymbol{\sigma})\}]^2. \tag{107}$$

We thus consistently retrieve for the Gaussian contangle (or, equivalently, the contangle, as they coincide in this specific case) the expression previously found for the mixed symmetric reductions of fully symmetric three-mode pure states, equation (99).

To our aims, it is useful here to provide a compact, operative definition of the Gaussian contangle for $1 \times N$ bipartite Gaussian states, based on the evaluation of Gaussian entanglement measures in section 4.5.2. If $\boldsymbol{\sigma}_{i|j}$ is the CM of a (generally mixed) bipartite Gaussian state where subsystem $\mathcal{S}_i$ comprises one mode only, then the Gaussian contangle G_τ can be computed as

$$G_\tau(\boldsymbol{\sigma}_{i|j}) \equiv G_\tau\left(\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}}\right) = g\left[m_{i|j}^2\right], \qquad g[x] = \operatorname{arcsinh}^2[\sqrt{x-1}]. \tag{108}$$

Here, $\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}}$ corresponds to a pure Gaussian state and $m_{i|j} \equiv m\left(\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}}\right) = \sqrt{\operatorname{Det} \boldsymbol{\sigma}_i^{\mathrm{opt}}} = \sqrt{\operatorname{Det} \boldsymbol{\sigma}_j^{\mathrm{opt}}}$, with $\boldsymbol{\sigma}_{i(j)}^{\mathrm{opt}}$ being the reduced CM of subsystem $\mathcal{S}_i$ ($\mathcal{S}_j$) obtained by tracing over the degrees of freedom of subsystem $\mathcal{S}_j$ ($\mathcal{S}_i$). The CM $\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}}$ denotes the pure bipartite Gaussian state which minimizes $m\left(\boldsymbol{\sigma}_{i|j}^p\right)$ among all pure-state CMs $\boldsymbol{\sigma}_{i|j}^p$ such that $\boldsymbol{\sigma}_{i|j}^p \leqslant \boldsymbol{\sigma}_{i|j}$. If $\boldsymbol{\sigma}_{i|j}$ is a pure state, then $\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}} = \boldsymbol{\sigma}_{i|j}$, while for a mixed Gaussian state equation (108) is mathematically equivalent to constructing the Gaussian convex roof. For a separable state $m\left(\boldsymbol{\sigma}_{i|j}^{\mathrm{opt}}\right) = 1$. Here, we have implicitly used the fact that the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partial transpose of a pure $1 \times N$ Gaussian state $\boldsymbol{\sigma}_{i|j}^p$ is equal to $\tilde{\nu}_- = \sqrt{\operatorname{Det} \boldsymbol{\sigma}_i} - \sqrt{\operatorname{Det} \boldsymbol{\sigma}_i - 1}$, as follows by recalling that the $1 \times N$ entanglement is equivalent to a 1×1 entanglement by virtue of the phase-space Schmidt decomposition (see section 3.4.1) and by exploiting equation (80) with $\Delta = 2$, $\mu = 1$ and $\mu_1 = \mu_2 \equiv 1/\sqrt{\operatorname{Det} \boldsymbol{\sigma}_i}$.

The Gaussian contangle G_τ, like all members of the family of measures of entanglement (see section 4.5.2) is completely equivalent to the Gaussian entanglement of formation [106], which quantifies the cost of creating a given mixed Gaussian state out of an ensemble of pure, entangled Gaussian states.

Gaussian contangle. Analogously, for a $1 \times N$ bipartition associated with a pure Gaussian state $\varrho_{A|B}^p$ with $\mathcal{S}_A = \mathcal{S}_1$ (a subsystem of a single mode) and $\mathcal{S}_B = \mathcal{S}_2 \ldots \mathcal{S}_N$, we define the following quantity:

$$\tau_G\left(\varrho_{A|B}^p\right) = \mathcal{N}^2\left(\varrho_{A|B}^p\right). \tag{109}$$

Here, $\mathcal{N}(\varrho)$ is the negativity, equation (63), of the Gaussian state ϱ. The functional τ_G, like the negativity $\mathcal{N}$, vanishes on separable states and does not increase under LOCC, i.e., it is a proper measure of pure-state bipartite entanglement. It can be naturally extended to mixed Gaussian states $\rho_{A|B}$ via the convex-roof construction

$$\tau_G(\varrho_{A|B}) = \inf_{\{p_i,\varrho_i^{(p)}\}} \sum_i p_i \tau_G\big(\varrho_i^p\big), \tag{110}$$

where the infimum is taken over all convex decompositions of $\varrho_{A|B}$ in terms of pure *Gaussian* states ϱ_i^p: $\rho_{A|B} = \sum_i p_i \varrho_i^p$. By virtue of the Gaussian convex-roof construction, the Gaussian entanglement measure τ_G, equation (110), is an entanglement monotone under Gaussian LOCC (see section 4.5.2). Henceforth, given an arbitrary N-mode Gaussian state $\varrho_{\mathcal{S}_1|\mathcal{S}_2\ldots\mathcal{S}_N}$, we refer to τ_G, equation (110), as the *Gaussian tangle* [122]. Obviously, in terms of CMs, the analogous of the definition (106) is valid for the Gaussian tangle as well, yielding it computable like the contangle in equation (108). Namely, exploiting equation (67), one finds

$$\tau_G(\sigma_{i|j}) \equiv \tau_G\big(\sigma_{i|j}^{\rm opt}\big) = w\big[m_{i|j}^2\big], \qquad w[x] = \tfrac{1}{4}(\sqrt{x-1}+\sqrt{x}-1)^2, \tag{111}$$

where we refer to the discussion immediately after equation (108) for the definition of the quantities involved in equation (111).

We will now proceed to review the structure of entanglement sharing in Gaussian states and the expressions of the monogamy constraints on its distribution. We remark that, being the (squared) negativity a monotonic and convex function of the (squared) logarithmic negativity, see equation (102), the validity of any monogamy constraint on distributed Gaussian entanglement using as a measure of entanglement the 'Gaussian tangle' is *implied* by the proof of the corresponding monogamy inequality obtained using the 'Gaussian contangle'. For this reason, when possible, we will always employ as a preferred choice the primitive entanglement monotone, represented by the (Gaussian) contangle [121] (which could be generally referred to as a 'logarithmic' tangle in quantum systems of arbitrary dimension).

7.3. Monogamy inequality for all Gaussian states

We are now in the position to recall a collection of recent results concerning the monogamy of distributed Gaussian entanglement in multimode Gaussian states.

In the broadest setting, we want to investigate whether a monogamy inequality like in equation (97) holds in the general case of Gaussian states with an arbitrary number N of modes. Considering a Gaussian state distributed among N parties (each owning a single mode), the monogamy constraint on distributed entanglement can be written as

$$E^{\mathcal{S}_i|(\mathcal{S}_1\ldots\mathcal{S}_{i-1}\mathcal{S}_{i+1}\ldots\mathcal{S}_N)} \geqslant \sum_{j\neq i}^{N} E^{\mathcal{S}_i|\mathcal{S}_j} \tag{112}$$

where the global system is multipartitioned in subsystems $\mathcal{S}_k$ $(k = 1, \ldots, N)$, each owned by a respective party, and E is a proper measure of bipartite entanglement. The corresponding general monogamy inequality is known to hold for qubit systems [134].

The left-hand side of inequality (112) quantifies the bipartite entanglement between a probe subsystem $\mathcal{S}_i$ and the remaining subsystems taken as a whole. The right-hand side quantifies the total bipartite entanglement between $\mathcal{S}_i$ and each one of the other subsystems $\mathcal{S}_{j\neq i}$ in the respective reduced states. The nonnegative difference between these two entanglements, minimized over all choices of the probe subsystem, is referred to as the *residual multipartite entanglement*. It quantifies the purely quantum correlations that are not encoded in pairwise form, so it includes all manifestations of genuine K-partite entanglement, involving K subsystems (modes) at a time, with $2 < K \leqslant N$. The study of entanglement sharing and monogamy constraints thus offers a natural framework to interpret and quantify entanglement in multipartite quantum systems [136].

With these premises, we have proven that the (Gaussian) contangle (and the Gaussian tangle, as an implication) is monogamous in fully symmetric Gaussian states of N modes [121]. In general, we have proven the Gaussian tangle to satisfy inequality (112) in *all* N-mode Gaussian states [122]. A full analytical proof of the monogamy inequality for the contangle in all Gaussian states beyond the symmetry is currently lacking; however, numerical evidence obtained for randomly generated nonsymmetric four-mode Gaussian states strongly supports the conjecture that the monogamy inequality be true for all multimode Gaussian states, also using the (Gaussian) contangle as a measure of bipartite entanglement [121]. Remarkably, for all (generally nonsymmetric) three-mode Gaussian states the (Gaussian) contangle has been proven to be monogamous, leading in particular to a proper measure of tripartite entanglement in terms of residual contangle: the analysis of distributed entanglement in the special instance of three-mode Gaussian states, with all the resulting implications, is postponed to the next section.

Let us restate again the main result of this section: *The Gaussian tangle τ_G, an entanglement monotone under Gaussian LOCC, is monogamous for all, pure and mixed*, N-mode Gaussian states distributed among N *parties, each owning a single mode* [122].

7.4. Discussion

The monogamy constraints on entanglement sharing are essential for the security of CV quantum cryptographic schemes [138, 139], because they limit the information that might be extracted from the secret key by a malicious eavesdropper. Monogamy is useful as well in investigating the range of correlations in Gaussian matrix-product states of harmonic rings [30, 140] and in understanding the entanglement frustration occurring in ground states of many-body harmonic lattice systems [141], which may be now extended to arbitrary states beyond symmetry constraints.

At a fundamental level, having established the monogamy property for all Gaussian states paves the way to a proper quantification of genuine multipartite entanglement in CV systems in terms of the residual distributed entanglement. In this respect, the intriguing question arises whether a *stronger* monogamy constraint exists on the distribution of entanglement in many-body systems, which imposes a physical trade-off on the sharing of both bipartite and genuine multipartite quantum correlations.

It would be important to understand whether the inequality (112) holds as well for discrete-variable qu*d*its ($2 < d < \infty$), interpolating between qubits and CV systems [136]. If this were the case, the (convex-roof extended) squared negativity, which coincides with the tangle for arbitrary states of qubits and with the Gaussian tangle for Gaussian states of CV systems, would qualify as a universal *bona fide*, dimension-independent quantifier of entanglement sharing in all multipartite quantum systems. At present, this must be considered as a completely open problem.

8. Multipartite Gaussian entanglement: quantification and promiscuous sharing structure

This section is mainly devoted to the characterization of entanglement in the simplest multipartite CV setting, namely that of three-mode Gaussian states.

To begin with, let us set the notation and review the known results about three-mode Gaussian states of CV systems. We will refer to the three modes under exam as modes 1, 2 and 3. The 2×2 submatrices that form the CM $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_{123}$ of a three-mode Gaussian state are defined according to equation (19), whereas the 4×4 CMs of the reduced two-mode Gaussian states of modes i and j will be denoted by $\boldsymbol{\sigma}_{ij}$. Likewise, the local (two-mode) seralian invariants Δ_{ij}, equation (32), will be specified by the labels i and j of the modes they refer to, while, to avoid any confusion, the three-mode (global) seralian symplectic invariant will be denoted by $\Delta \equiv \Delta_{123}$. Let us recall the uncertainty relation, equation (73), for two-mode Gaussian states,

$$\Delta_{ij} - \mathrm{Det}\,\boldsymbol{\sigma}_{ij} \leqslant 1. \qquad (113)$$

8.1. Separability classes for three-mode Gaussian states

As it is clear from the discussion of section 4.4.1, a complete *qualitative* characterization of the entanglement of three-mode Gaussian state is possible because the PPT criterion is necessary and sufficient for their separability under *any*, partial or global (i.e. 1×1 or 1×2), bipartition of the modes. This has led to an exhaustive classification of three-mode Gaussian states in five distinct separability classes [142]. These classes take into account the fact that the modes 1, 2 and 3 allow for three distinct global bipartitions:

- *Class 1*: states not separable under all the three possible bipartitions $i \times (jk)$ of the modes (fully inseparable states, possessing genuine multipartite entanglement).
- *Class 2*: states separable under only one of the three possible bipartitions (one-mode biseparable states).
- *Class 3*: states separable under only two of the three possible bipartitions (two-mode biseparable states).
- *Class 4*: states separable under all the three possible bipartitions, but impossible to write as a convex sum of tripartite products of pure one-mode states (three-mode biseparable states).
- *Class 5*: states that are separable under all the three possible bipartitions and can be written as a convex sum of tripartite products of pure one-mode states (fully separable states).

Note that Classes 4 and 5 cannot be distinguished by partial transposition of any of the three modes (which is positive for both classes). States in Class 4 stand therefore as nontrivial examples of tripartite entangled states of CV systems with positive partial transpose [142]. It is well known that entangled states with positive partial transpose possess *bound entanglement*, that is, entanglement that cannot be distilled by means of LOCC.

8.2. Residual contangle as genuine tripartite entanglement monotone

We have proven in [121] that all three-mode Gaussian states satisfy the CKW monogamy inequality (97), using the (Gaussian) contangle equation (108) to quantify bipartite entanglement.

The sharing constraint naturally leads to the definition of the *residual contangle* as a quantifier of genuine tripartite entanglement in three-mode Gaussian states, much in the same way as in systems of three qubits [133]. However, at variance with the three-qubit case (where the residual tangle of pure states is invariant under qubit permutations), here the residual contangle is partition dependent according to the choice of the probe mode, with the obvious exception of the fully symmetric states. A *bona fide* quantification of tripartite entanglement is then provided by the *minimum* residual contangle [121]

$$E_\tau^{i|j|k} \equiv \min_{(i,j,k)} \left[E_\tau^{i|(jk)} - E_\tau^{i|j} - E_\tau^{i|k}\right], \qquad (114)$$

where the symbol (i, j, k) denotes all the permutations of the three-mode indices. This definition ensures that $E_\tau^{i|j|k}$ is invariant under all permutations of the modes and is thus a genuine three-way property of any three-mode Gaussian state. We can adopt an analogous definition for the minimum residual Gaussian contangle $G_\tau^{\rm res}$, sometimes referred to as *arravogliament* [121, 131, 143]:

$$G_\tau^{\rm res} \equiv G_\tau^{i|j|k} \equiv \min_{(i,j,k)} \left[G_\tau^{i|(jk)} - G_\tau^{i|j} - G_\tau^{i|k}\right]. \tag{115}$$

One can verify that

$$\left(G_\tau^{i|(jk)} - G_\tau^{i|k}\right) - \left(G_\tau^{j|(ik)} - G_\tau^{j|k}\right) \geqslant 0 \tag{116}$$

if and only if $a_i \geqslant a_j$, and therefore the absolute minimum in equation (114) is attained by the decomposition realized with respect to the reference mode l of smallest local mixedness a_l, i.e. for the single-mode reduced state with CM of smallest determinant (corresponding to the largest local purity μ_l).

A crucial requirement for the residual (Gaussian) contangle, equation (115), to be a proper measure of tripartite entanglement is that it should be nonincreasing under (Gaussian) LOCC. The monotonicity of the residual tangle was proven for three-qubit pure states in [144]. In the CV setting we have proven that for pure three-mode Gaussian states $G_\tau^{\rm res}$ is an entanglement monotone under tripartite Gaussian LOCC and that it is nonincreasing even under probabilistic operations, which is a stronger property than being only monotone on average [121]. Therefore, the residual Gaussian contangle $G_\tau^{\rm res}$ is a proper and computable measure of genuine multipartite (specifically, tripartite) entanglement in three-mode Gaussian states.

8.3. Standard form and tripartite entanglement of pure three-mode Gaussian states

Here, we apply the above-defined measure of tripartite entanglement to the relevant instance of *pure* three-mode Gaussian states. We begin by recalling their structural properties.

8.3.1. Symplectic properties of three-mode covariance matrices. The CM $\boldsymbol{\sigma}$ of a pure three-mode Gaussian state is characterized by

$$\mathrm{Det}\,\boldsymbol{\sigma} = 1, \qquad \Delta = 3. \tag{117}$$

The purity constraint requires the local entropic measures of any 1×2-mode bipartitions to be equal:

$$\mathrm{Det}\,\boldsymbol{\sigma}_{ij} = \mathrm{Det}\,\boldsymbol{\sigma}_k, \tag{118}$$

with i, j and k being different from each other. This general, well-known property of the bipartitions of pure states may be easily proven resorting to the Schmidt decomposition (see section 3.4.1).

A first consequence of equations (117) and (118) is rather remarkable. Combining such equations one easily obtains

$$(\Delta_{12} - \mathrm{Det}\,\boldsymbol{\sigma}_{12}) + (\Delta_{13} - \mathrm{Det}\,\boldsymbol{\sigma}_{13}) + (\Delta_{23} - \mathrm{Det}\,\boldsymbol{\sigma}_{23}) = 3,$$

which, together with inequality (113), implies

$$\Delta_{ij} = \mathrm{Det}\,\boldsymbol{\sigma}_{ij} + 1, \qquad \forall i, j : i \neq j. \tag{119}$$

The last equation shows that any reduced two-mode state of a pure three-mode Gaussian state saturates the partial uncertainty relation, equation (113). The states endowed with such a partial minimal uncertainty (namely, with their smallest symplectic eigenvalue equal to 1) are states of minimal negativity for given global and local purities, alias GLEMS (Gaussian least entangled mixed states) [37, 104], introduced in section 5.3.

This is relevant considering that the standard-form CM of Gaussian states is completely determined by their global and local invariants. Therefore, because of equation (118), the entanglement between any pair of modes embedded in a three-mode pure Gaussian state is fully determined by the local invariants $\mathrm{Det}\,\boldsymbol{\sigma}_l$, for $l = 1, 2, 3$, whatever proper measure we choose to quantify it. Furthermore, the entanglement of a $\boldsymbol{\sigma}_i|\boldsymbol{\sigma}_{jk}$ bipartition of a pure three-mode state is determined by the entropy of one of the reduced states that is, once again, by the quantity $\mathrm{Det}\,\boldsymbol{\sigma}_i$. Thus, *the three local symplectic invariants* $\mathrm{Det}\,\boldsymbol{\sigma}_1$, $\mathrm{Det}\,\boldsymbol{\sigma}_2$ *and* $\mathrm{Det}\,\boldsymbol{\sigma}_3$ *fully determine the entanglement of any bipartition of a pure three-mode Gaussian state*. We will show that they suffice to determine as well the genuine tripartite entanglement encoded in the state [131].

For ease of notation, in the following we will denote by a_l the local single-mode symplectic eigenvalues associated with mode l with CM $\boldsymbol{\sigma}_l$:

$$a_l \equiv \sqrt{\mathrm{Det}\,\boldsymbol{\sigma}_l}. \tag{120}$$

Equation (46) shows that the quantities a_l are simply related to the purities of the reduced single-mode states, the local purities μ_l, by the relation

$$\mu_l = \frac{1}{a_l}. \tag{121}$$

Since the set $\{a_l\}$ fully determines the entanglement of any of the 1×2 and 1×1 bipartitions of the state, it is important to determine the range of the allowed values for such quantities. This will provide a complete quantitative characterization of the entanglement of three-mode pure Gaussian states.

Such an analysis has been performed in [131]. Defining the parameters

$$a'_l \equiv a_l - 1, \tag{122}$$

one has that the following permutation-invariant triangular inequality

$$|a'_i - a'_j| \leqslant a'_k \leqslant a'_i + a'_j, \tag{123}$$

holds as the only condition, together with the positivity of each a'_l, which fully characterizes the local symplectic eigenvalues of the CM of three-mode pure Gaussian states. All standard forms of pure three-mode Gaussian states and in particular, remarkably, all the possible values of the negativities (section 4.5.1) and/or of the Gaussian entanglement measures (section 4.5.2) between *any* pair of subsystems, can be determined by letting a'_1, a'_2 and a'_3 vary in their range of allowed values. Let us remark that equation (123) qualifies itself as an entropic inequality, as the quantities $\{a'_j\}$ are closely related to the purities and to the von Neumann entropies of the single-mode reduced states. In particular, the von Neumann entropies S_{Vj} of the reduced states are given by $S_{Vj} = f(a'_j{+}1) = f(a_j)$, where the increasing convex entropic function $f(x)$ has been defined in equation (48). Now, inequality (123) is strikingly analogous to the well-known triangle (Araki–Lieb) and subadditivity inequalities for the von Neumann entropy (holding for general systems [36, 145]), which in our case read

$$|f(a_i) - f(a_j)| \leqslant f(a_k) \leqslant f(a_i) + f(a_j). \tag{124}$$

However, the condition imposed by equation (123) is strictly *stronger* than the generally holding inequalities (124) for the von Neumann entropy applied to pure quantum states.

We recall that the form of the CM of any Gaussian state can be simplified through local (unitary) symplectic operations, that therefore do not affect the entanglement or mixedness properties of the state, belonging to $Sp_{2,\mathbb{R}}^{\oplus N}$ [30, 106, 146]. Such reductions of the CMs are called 'standard forms', as introduced in section 3.4. For the sake of clarity, let us write the explicit standard-form CM of a generic *pure* three-mode Gaussian state [131],

$$\boldsymbol{\sigma}^p_{sf} = \begin{pmatrix} a_1 & 0 & e^+_{12} & 0 & e^+_{13} & 0 \\ 0 & a_1 & 0 & e^-_{12} & 0 & e^-_{13} \\ e^+_{12} & 0 & a_2 & 0 & e^+_{23} & 0 \\ 0 & e^-_{12} & 0 & a_2 & 0 & e^-_{23} \\ e^+_{13} & 0 & e^+_{23} & 0 & a_3 & 0 \\ 0 & e^-_{13} & 0 & e^-_{23} & 0 & a_3 \end{pmatrix}, \tag{125}$$

with

$$e^{\pm}_{ij} \equiv \frac{1}{4\sqrt{a_i a_j}}\Big\{\sqrt{[(a_i-a_j)^2-(a_k-1)^2][(a_i-a_j)^2-(a_k+1)^2]} \pm \sqrt{[(a_i+a_j)^2-(a_k-1)^2][(a_i+a_j)^2-(a_k+1)^2]}\Big\}. \tag{126}$$

Let us stress that, although useful in actual calculations, the use of CMs in standard form does not entail any loss of generality, because all the results concerning the characterization of bipartite and tripartite entanglement do not depend on the choice of the specific form of the CMs, but only on invariant quantities, such as the global and local symplectic invariants.

8.3.2. Residual contangle of fully inseparable three-mode Gaussian states. We can now analyse separability and entanglement in more detail. From our study it turns out that, regarding the classification of section 8.1 [142], pure three-mode Gaussian states may belong either to Class 5, in which case they reduce to the global three-mode vacuum, or to Class 2, reducing to the uncorrelated product of a single-mode vacuum and of a two-mode squeezed state or to Class 1 (fully inseparable state). No two-mode or three-mode biseparable pure three-mode Gaussian states are allowed.

Let us now describe the complete procedure to determine the genuine tripartite entanglement in a *pure* three-mode Gaussian state with a completely general (not necessarily in standard form) CM $\boldsymbol{\sigma}^p$ belonging to Class 1, as presented in [131].

(i) *Determine the local purities.* The state is globally pure ($\mathrm{Det}\,\boldsymbol{\sigma}^p = 1$). The only quantities needed for the computation of the tripartite entanglement are therefore the three local mixednesses a_l, defined by equation (120), of the single-mode reduced states $\boldsymbol{\sigma}_l$, $l = 1, 2, 3$ (see equation (19)). Note that the global CM $\boldsymbol{\sigma}^p$ needs not to be in the standard form of equation (125), as the single-mode determinants are local symplectic invariants. From an experimental point of view, the parameters a_l can be extracted from the CM using the homodyne tomographic reconstruction of the state [117]; or they can be directly measured with the aid of single-photon detectors [111, 112].

(ii) *Find the minimum.* From equation (116), the minimum in definition (115) of the residual Gaussian contangle G^{res}_{τ} is attained in the partition where the bipartite entanglements are decomposed choosing as probe mode l the one in the single-mode reduced state of smallest local mixedness $a_l \equiv a_{\min}$.

(iii) *Check range and compute.* Given the mode with smallest local mixedness $a_{\min}$ (say, for instance, mode 1) and the parameters s and d defined by

$$s = \frac{a_2 + a_3}{2}, \tag{127}$$

$$d = \frac{a_2 - a_3}{2}, \tag{128}$$

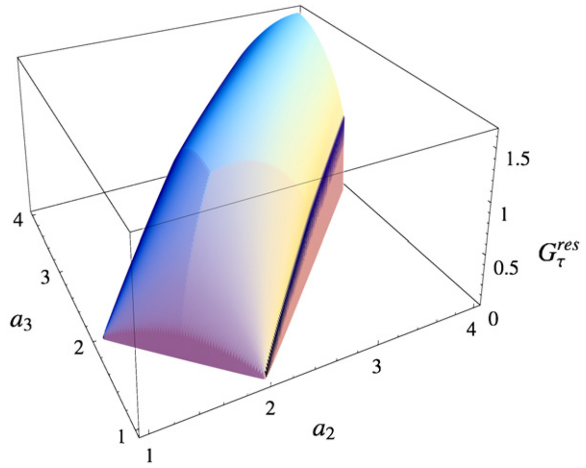


Figure 7. Three-dimensional plot of the residual Gaussian contangle $G_\tau^{\text{res}}(\sigma^p)$ in pure three-mode Gaussian states σ^p, determined by the three local mixednesses $a_l, l = 1, 2, 3$. One of the local mixednesses is kept fixed ($a_1 = 2$). The remaining ones vary constrained by the triangle inequality (123). The explicit expression of G_τ^{res} is given by equation (129). See the text for further details.

if $a_{\min} = 1$ then mode 1 is uncorrelated from the others: $G_\tau^{\text{res}} = 0$. If, instead, $a_{\min} > 1$ then

$$G_\tau^{\text{res}}(\boldsymbol{\sigma}^p) = \text{arcsinh}^2\left[\sqrt{a_{\min}^2 - 1}\right] - Q(a_{\min}, s, d), \tag{129}$$

with $Q \equiv G_\tau^{1|2} + G_\tau^{1|3}$ defined by

$$Q(a, s, d) = \text{arcsinh}^2\left[\sqrt{m^2(a, s, d) - 1}\right] + \text{arcsinh}^2\left[\sqrt{m^2(a, s, -d) - 1}\right], \tag{130}$$

where $m = m_-$ if $D \leqslant 0$, and $m = m_+$ otherwise (one has $m_+ = m_-$ for $D = 0$). Here,

$$\begin{aligned}
m_- &= \frac{|k_-|}{(s-d)^2 - 1}, \\
m_+ &= \frac{\sqrt{2[2a^2(1 + 2s^2 + 2d^2) - (4s^2 - 1)(4d^2 - 1) - a^4 - \sqrt{\delta}]}}{4(s-d)}, \\
D &= 2(s-d) - \sqrt{2[k_-^2 + 2k_+ + |k_-|(k_-^2 + 8k_+)^{1/2}]/k_+}, \\
k_\pm &= a^2 \pm (s+d)^2, \\
\delta &= (a - 2d - 1)(a - 2d + 1)(a + 2d - 1)(a + 2d + 1) \\
&\quad (a - 2s - 1)(a - 2s + 1)(a + 2s - 1)(a + 2s + 1).
\end{aligned}$$

Note (we omitted the explicit dependence for brevity) that each quantity in equation (131) is a function of (a, s, d). Therefore, to evaluate the second term in equation (130) each d in equation (131) must be replaced by $-d$. Note also that if $d < -\left(a_{\min}^2 - 1\right)/4s$ then $G_\tau^{1|2} = 0$. Instead, if $d > \left(a_{\min}^2 - 1\right)/4s$ then $G_\tau^{1|3} = 0$. Otherwise, all terms in equation (115) are nonvanishing.

The residual Gaussian contangle equation (115) in generic pure three-mode Gaussian states is plotted in figure 7 as a function of a_2 and a_3, at constant $a_1 = 2$. For fixed a_1, it is interesting to notice that G_τ^{res} is maximal for $a_2 = a_3$, i.e. for bisymmetric states. Note also how the residual Gaussian contangle of these bisymmetric pure states has a cusp for $a_1 = a_2 = a_3$. In fact, from equation (116), for $a_2 = a_3 < a_1$ the minimum in equation (115) is attained decomposing with respect to one of the two modes 2 or 3 (the result is the same by symmetry), while for $a_2 = a_3 > a_1$ mode 1 becomes the probe mode.

8.3.3. Residual contangle and distillability of mixed states. For generic *mixed* three-mode Gaussian states, a quite cumbersome analytical expression for the $1|2$ and $1|3$ Gaussian contangles may be written, which explicitly solves the minimization over the angle θ in equation (90). On the other hand, the optimization appearing in the computation of the $1|(23)$ bipartite Gaussian contangle (see equation (108)) has to be solved only numerically. However, exploiting techniques like the unitary localization of entanglement described in section 6, and results like that of equation (107), closed expressions for the residual Gaussian contangle can be found as well in relevant classes of mixed three-mode Gaussian states endowed with some symmetry constraints. Interesting examples of these states and the investigation of their physical properties are discussed in [131, 143].

As an additional remark, let us recall that, although the entanglement of Gaussian states is always distillable with respect to $1 \times N$ bipartitions [80] (see section 4.4.1), they can exhibit bound entanglement in $1 \times 1 \times 1$ tripartitions [142]. In this case, the residual Gaussian contangle cannot detect tripartite PPT entangled states. For example, the residual Gaussian contangle in three-mode biseparable Gaussian states (Class 4 of [142]) is always zero, because those bound entangled states are separable with respect to all 1×2 bipartitions of the modes. In this sense we can correctly regard the residual Gaussian contangle as an estimator of *distillable* tripartite entanglement, being strictly nonzero only on fully inseparable three-mode Gaussian states (Class 1 in the classification of section 8.1).

8.4. Sharing structure of multipartite entanglement: promiscuous Gaussian states

We are now in the position to review the sharing structure of CV entanglement in three-mode Gaussian states by taking the residual Gaussian contangle as a measure of tripartite entanglement, in analogy with the study done for three qubits [144] using the residual tangle [133].

The first task we face is that of identifying the three-mode analogues of the two inequivalent classes of fully inseparable three-qubit states, the GHZ state [147]

$$|\psi_{\mathrm{GHZ}}\rangle = \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{131}$$

and the W state [144]

$$|\psi_W\rangle = \tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{132}$$

These states are both pure and fully symmetric, i.e. invariant under the exchange of any two qubits. On the one hand, the GHZ state possesses maximal tripartite entanglement, quantified by the residual tangle [133, 144], with zero couplewise entanglement in any reduced state of two qubits reductions. Therefore, its entanglement is very fragile against the loss of one or more subsystems. On the other hand, the W state contains the maximal two-party entanglement in any reduced state of two qubits [144] and is thus maximally robust against decoherence, while its tripartite residual tangle vanishes[8].

8.4.1. CV finite-squeezing GHZ/W states. To define the CV counterparts of the three-qubit states $|\psi_{\mathrm{GHZ}}\rangle$ and $|\psi_W\rangle$, one must start from the fully symmetric (generally mixed) three-mode CM $\boldsymbol{\sigma}_s$ of the form $\boldsymbol{\sigma}_{\alpha^3}$, equation (51). Surprisingly enough, in symmetric three-mode Gaussian states, if one aims at maximizing, at given single-mode mixedness $a \equiv \sqrt{\mathrm{Det}\,\boldsymbol{\alpha}}$,

[8] The full inseparability of the W state can however be detected by the so-called 'Schmidt measure' [148] or by introducing for qubits also a monogamy inequality in terms of the negativity [149].

either the bipartite entanglement $G_\tau^{i|j}$ in any two-mode reduced state (i.e. aiming at the CV W-like state), or the genuine tripartite entanglement $G_\tau^{\rm res}$ (i.e. aiming at the CV GHZ-like state), one finds the same, unique family of states. They are exactly the *pure*, fully symmetric three-mode Gaussian states (three-mode squeezed states) with CM $\boldsymbol{\sigma}_s^p$ of the form $\boldsymbol{\sigma}_{\alpha^3}$, equation (51), with $\boldsymbol{\alpha} = a\mathbb{1}_2$, $\boldsymbol{\varepsilon} = \mathrm{diag}\{e^+, e^-\}$ and

$$e^{\pm} = \frac{a^2 - 1 \pm \sqrt{(a^2-1)(9a^2-1)}}{4a}, \tag{133}$$

where we have used equation (94) ensuring the global purity of the state. In general, we have studied the entanglement scaling in fully symmetric (pure) Gaussian states by means of the unitary localization in section 6.2. It is in order to mention that these states were previously known in the literature as CV 'GHZ-type' states [90, 128], as in the limit of infinite squeezing ($a \to \infty$), they approach the proper (unnormalizable) continuous-variable GHZ state $\int \mathrm{d}x |x, x, x\rangle$, a simultaneous eigenstate of total momentum $\hat{p}_1 + \hat{p}_2 + \hat{p}_3$ and of all relative positions $\hat{q}_i - \hat{q}_j$ $(i, j = 1, 2, 3)$, with zero eigenvalues [150].

For any finite squeezing (equivalently, any finite local mixedness a), however, the above entanglement sharing study leads ourselves to re-baptize these states as 'CV GHZ/W states' [121, 131, 143], and denote their CM by $\boldsymbol{\sigma}_s^{\mathrm{GHZ}/W}$.

The residual Gaussian contangle of GHZ/W states of finite squeezing takes the simple form [121]

$$G_\tau^{\rm res}\left(\boldsymbol{\sigma}_s^{\mathrm{GHZ}/W}\right) = \mathrm{arcsinh}^2\left[\sqrt{a^2-1}\right] - \frac{1}{2}\log^2\left[\frac{3a^2 - 1 - \sqrt{9a^4 - 10a^2 + 1}}{2}\right]. \tag{134}$$

It is straightforward to see that $G_\tau^{\rm res}\left(\boldsymbol{\sigma}_s^{\mathrm{GHZ}/W}\right)$ is nonvanishing as soon as $a > 1$. Therefore, the GHZ/W states belong to the class of fully inseparable three-mode states [90, 128, 142, 151] (Class 1, see section 8.1). We finally recall that in a GHZ/W state the residual Gaussian contangle $G_\tau^{\rm res}$, equation (115), coincides with the true residual contangle $E_\tau^{1|2|3}$, equation (114). This property clearly holds because the Gaussian pure-state decomposition is the optimal one in every bipartition, due to the fact that the global three-mode state is pure and the reduced two-mode states are symmetric (see section 5.2.2).

The peculiar nature of entanglement sharing in CV GHZ/W states is further confirmed by the following observation. If one requires maximization of the 1×2 bipartite Gaussian contangle $G_\tau^{i|(jk)}$ under the constraint of separability of all the reduced two-mode states, one finds a class of symmetric mixed states (T states [121, 131, 143]) whose residual contangle is strictly smaller than the one of the GHZ/W states equation (134) for any fixed value of the local mixedness a, that is, for any fixed value of the only parameter (operationally related to the squeezing of each single mode) that completely determines the CMs of both families of states up to local unitary operations.

8.4.2. Promiscuous continuous-variable entanglement sharing. The above results lead to the conclusion that in symmetric three-mode Gaussian states, when there is no bipartite entanglement in the two-mode reduced states (like in T states), the genuine tripartite entanglement is not enhanced, but frustrated. More than that, if there are maximal quantum correlations in a three-party relation, like in GHZ/W states, then the two-mode reduced states of any pair of modes are maximally entangled mixed states.

These findings establish the *promiscuous* nature of CV entanglement sharing in symmetric Gaussian states [121]. Being associated with degrees of freedom with continuous spectra, states of CV systems need not saturate the CKW inequality to achieve maximum couplewise correlations (as it was instead the case for W states of qubits, equation (132)). In fact, without violating the monogamy constraint in equation (97), pure symmetric three-mode Gaussian states are maximally three-way entangled and, at the same time, possess the maximum possible entanglement between any pair of modes in the corresponding two-mode reduced states. The notion of 'promiscuity' basically means that bipartite and genuine multipartite (in this case tripartite) entanglement are increasing functions of each other, while typically in low-dimensional systems like qubits only the opposite behaviour is compatible with monogamy [136]. The promiscuity of entanglement in three-mode GHZ/W states is, however, *partial*. Namely they exhibit, with increasing squeezing, unlimited tripartite entanglement (diverging in the limit $a \to \infty$) and nonzero, accordingly increasing bipartite entanglement between any two modes, which nevertheless stays finite even for infinite squeezing. Precisely, from equation (134), it saturates to the value

$$G_\tau^{i|j}\left(\sigma_s^{\mathrm{GHZ}/W}, a \to \infty\right) = \frac{\log^2 3}{4} \approx 0.3. \tag{135}$$

In Gaussian states of CV systems with more than three modes, entanglement can indeed be distributed in an *infinitely* promiscuous way [132], as we will briefly discuss in the following.

More remarks are in order concerning the tripartite case. The structure of entanglement in GHZ/W states is such that, while being maximally three-party entangled, they are also maximally robust against the loss of one of the modes. This preselects GHZ/W states also as optimal candidates for carrying quantum information through a lossy channel, being intrinsically less sensitive to decoherence effects. They have been in fact proven to be maximally robust against environmental noise among all three-mode Gaussian states [131].

It is natural to question whether *all* three-mode Gaussian states are expected to exhibit a promiscuous entanglement sharing. Such a question is addressed in [143], by investigating the persistency of promiscuity against the lack of each of the two defining properties of GHZ/W states: full symmetry and global purity. One specifically finds that entanglement promiscuity survives under a quite strong amount of mixedness (up to an impurity of $1 - \mu \approx 0.8$), but is in general lost if the complete permutation-invariance is relaxed. Pure three-mode Gaussian states which are only bisymmetric and not fully symmetric (known as *basset hound states*) offer indeed a traditional, not promiscuous entanglement sharing, with bipartite and tripartite entanglement being competitors [143]. Therefore, in the tripartite Gaussian setting, '*promiscuity*' is a peculiar consequence not of the global purity, but of the complete symmetry under modes-exchange. Beside frustrating the maximal entanglement between pairs of modes [141], symmetry also constrains the multipartite sharing of quantum correlations. In basset hound states, the separability of the reduced state of modes 2 and 3 prevents the three modes from having a strong genuine tripartite entanglement among them all, despite the heavy quantum correlations shared by the two couples of modes $1|2$ and $1|3$.

8.4.3. Unlimited promiscuity of entanglement in four-mode Gaussian states. The above argument on the origin of promiscuity does not hold in the case of Gaussian states with four and more modes, where relaxing the symmetry constraints may allow for an enhancement of the distributed entanglement promiscuity to an unlimited extent. In [132], we have introduced a class of pure four-mode Gaussian states which are not fully symmetric, but invariant under the double exchange of modes $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. They are defined as follows.

One starts with an uncorrelated state of four modes, each one initially in the vacuum of the respective Fock space, whose corresponding CM is the identity. One applies a two-mode squeezing transformation $S_{2,3}(s)$, equation (23), with squeezing s to modes 2 and 3, then two further two-mode squeezing transformations $S_{1,2}(a)$ and $S_{3,4}(a)$, with squeezing a, to the pairs of modes $\{1, 2\}$ and $\{3, 4\}$. The two last transformations serve the purpose of redistributing the original bipartite entanglement, created between modes 2 and 3 by the first two-mode squeezing operations, among all the four modes. For any value of the parameters s and a, the output is a pure four-mode Gaussian state with CM $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = S_{3,4}(a) S_{1,2}(a) S_{2,3}(s) S_{2,3}^{\mathsf{T}}(s) S_{1,2}^{\mathsf{T}}(a) S_{3,4}^{\mathsf{T}}(a). \tag{136}$$

Explicitly, $\boldsymbol{\sigma}$ is of the form equation (19) where

$$\begin{aligned}
\boldsymbol{\sigma}_1 &= \boldsymbol{\sigma}_4 = [\cosh^2(a) + \cosh(2s)\sinh^2(a)]\mathbb{1}_2,\\
\boldsymbol{\sigma}_2 &= \boldsymbol{\sigma}_3 = [\cosh(2s)\cosh^2(a) + \sinh^2(a)]\mathbb{1}_2,\\
\varepsilon_{1,2} &= \varepsilon_{3,4} = [\cosh^2(s)\sinh(2a)]Z_2,\\
\varepsilon_{1,3} &= \varepsilon_{2,4} = [\cosh(a)\sinh(a)\sinh(2s)]\mathbb{1}_2,\\
\varepsilon_{1,4} &= [\sinh^2(a)\sinh(2s)]Z_2,\\
\varepsilon_{2,3} &= [\cosh^2(a)\sinh(2s)]Z_2,
\end{aligned}$$

with $Z_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

By a proper investigation of the properties of distributed entanglement in the parametric state, equation (136), it can be shown that the entanglement between modes 1 and 2 is an unboundedly increasing function of the squeezing a; the same holds for the pair of modes 3 and 4. In addition, the block of modes (1, 2) is arbitrarily entangled with the block of modes (3, 4) as a function of s. On the other hand, one can demonstrate that in the same state an unlimited genuine four-partite entanglement is present, increasing arbitrarily with increasing a, and thus coexisting with (and being mutually enhanced by) the bipartite entanglement in two pairs of modes [132].

Such a simple example demonstrates that, when the quantum correlations arise among degrees of freedom spanning an infinite-dimensional space of states (characterized by unbounded mean energy), an accordingly infinite freedom is allowed for different forms of bipartite and multipartite entanglements. This phenomenon happens with no violation of the fundamental monogamy constraint that retains its general validity in quantum mechanics. In the CV instance the only effect of monogamy is to bound the divergence rates of the individual entanglement contributions as the squeezing parameters are increased. Within the restricted Hilbert space of four or more qubits, instead, an analogous entanglement structure between the single qubits is strictly forbidden.

This result opens interesting perspectives for the understanding and characterization of entanglement in multiparticle systems. Gaussian states with finite squeezing (finite mean energy) are somehow analogous to discrete systems with an effective dimension related to the amount of squeezing [1]. As the promiscuous entanglement sharing arises in Gaussian states by asymptotically increasing the squeezing to infinity, it is natural to expect that dimension-dependent families of states will exhibit an entanglement structure that becomes gradually more promiscuous with increasing dimension of the Hilbert space towards the CV limit. A proper investigation on systems of qu*d*its ($2 < d < \infty$) is therefore a necessary step in order to develop a complete picture of entanglement sharing in many-body systems [136]. This program has been initiated by establishing a sharp discrepancy between the two *extrema* in the ladder of Hilbert space dimensions: in the case of CV systems in the limit of infinite squeezing (infinite mean energy) entanglement has been proven infinitely more shareable than that of individual qubits [132]. This fact could prelude to implementations of quantum information protocols with CV systems that *cannot* be achieved and not even devised with qubit resources.

9. Conclusions and outlook

9.1. Entanglement in non-Gaussian states

The infinite-dimensional quantum world is obviously not confined to Gaussian states. In fact, some recent results demonstrate that the current trends in the theoretical understanding and experimental control of CV entanglement are strongly pushing towards the boundaries of the territory of Gaussian states and Gaussian operations.

The entanglement of Gaussian states cannot be increased (distilled) by Gaussian operations [83–85]. Similarly, for universal one-way quantum computation using Gaussian cluster states, a single-mode non-Gaussian measurement is required [152]. Moreover, a fundamental motivation for investigating entanglement in non-Gaussian states stems from the property of *extremality* of Gaussian states: it has been recently proved that they are the least entangled states among all states of CV systems with given second moments [153]. Experimentally, it has been recently demonstrated [154] that a two-mode squeezed Gaussian state can be 'degaussified' by coherent subtraction of a single photon, resulting in a mixed non-Gaussian state whose nonlocal properties and entanglement degree are enhanced (enabling a better efficiency for teleporting coherent states [155]). Theoretically, even the characterization of bipartite entanglement (let alone multipartite) in non-Gaussian states stands as a formidable task.

One immediate observation is that any two-mode state with a CM corresponding to an entangled Gaussian state is itself entangled [151]. Therefore, most of the results reviewed in this paper may serve to detect entanglement in a broader class of states of infinite-dimensional Hilbert spaces. They are, however, all sufficient conditions on entanglement based only on the second moments of the canonical operators. As such, for arbitrary non-Gaussian states, they are in general very inefficient (meaning that most entangled non-Gaussian states fail to be detected by these criteria). The description of non-Gaussian states requires indeed to consider high order statistical moments. It could then be expected that inseparability criteria for these states should involve high order correlations. Recently, some separability criteria based on hierarchies of conditions involving higher moments of the canonical operators have been introduced to provide a sharper detection of inseparability in generic non-Gaussian states.

A first step in this direction has been taken by Agarwal and Biswas, who have applied the method of partial transposition to the uncertainty relations for the Schwinger realizations of the $SU(2)$ and $SU(1,1)$ algebras [156]. This approach can be successfully used to detect entanglement in the two-mode entangled non-Gaussian state described by the wavefunction $\psi(x_1, x_2) = (2/\pi)^{1/2}(\gamma_1 x_1 + \gamma_2 x_2)\exp -\left(x_1^2 + x_2^2\right)/2$, with $|\gamma_1|^2 + |\gamma_2|^2 = 1$, and in the $SU(2)$ minimum-uncertainty states [156, 157]. The demonstration of criteria consisting in hierarchies of arbitrary order moments has been achieved preliminarily by Hillery and Zubairy [158] and definitively by Shchukin and Vogel [159–161]. In particular, Shchukin and Vogel have introduced an elegant and unifying approach, based on the PPT requirement, that allows us to derive, in the form of an infinite series of inequalities, a necessary and sufficient condition for the negativity of the partial transposition $\tilde{\varrho}$ of a bipartite quantum state ϱ. The Shchukin–Vogel (SV) criterion includes as special cases all the above-mentioned conditions (including the ones on second moments [13, 79] qualifying entanglement in Gaussian states), thus demonstrating the important role of PPT in building a strong criterion for the detection of entanglement.

Here, we briefly review the main features of the SV criterion first introduced by Shchukin and Vogel [159] and later analysed in mathematical detail by Miranowicz and Piani [160, 161] (see also [162]). Consider two bosonic modes A_1 and A_2 with the associated annihilation and creation operators, respectively $\hat{a}_1, \hat{a}_1^\dagger$ and $\hat{a}_2, \hat{a}_2{}^\dagger$. Shchukin and Vogel showed that every Hermitian operator $\hat{X}$ that acts on the Hilbert space of the two modes is nonnegative if and only if for any operator $\hat{f}$ whose normally ordered form exists, i.e., for any operator $\hat{f}$ that can be written in the form

$$\hat{f} = \sum_{n,m,k,l} c_{nmkl} \hat{a}_1^{\dagger n} \hat{a}_1^m \hat{a}_2^{\dagger k} \hat{a}_2^l, \tag{137}$$

it holds that

$$\mathrm{Tr}\,\hat{f}^{\dagger}\hat{f}\hat{X} \geqslant 0. \tag{138}$$

Applying the above result to the partially transposed matrix $\tilde{\rho}$ of a quantum state density matrix ρ, one finds that $\tilde{\rho}$ is nonpositive if and only if

$$\mathrm{Tr}[\tilde{\varrho}\hat{f}^{\dagger}\hat{f}] < 0. \tag{139}$$

Equation (139) is thus a necessary and sufficient condition for the negativity of the partially transposed density matrix (NPT). As such, the NPT criterion is in principle able to detect the bipartite entanglement of all CV two-mode states, pure or mixed, Gaussian or non-Gaussian, with the exception of the PPT bound entangled states.

Some specific applications to selected classes of non-Gaussian states have been recently discussed in the two-mode setting [163], and preliminary extensions of the SV NPT criterion to multimode and multipartite cases have been introduced and applied to distinguish between different classes of separability [164]. To this aim, entanglement witnesses are useful as well [89]. Another, inequivalent condition based on matrices of moments (*realignment criterion*) has also been recently introduced for the qualification of bipartite entanglement in non-Gaussian states (and in general mixed states of quantum systems in arbitrary dimension) [162]. We should mention a further interesting approach to non-Gaussian entanglement analysed by McHugh *et al* [165], who showed that the entanglement of multiphoton squeezed states is completely characterized by observing that with respect to a new set of modes those non-Gaussian states actually assume a Gaussian character.

The efficiency of some of the above-mentioned inseparability criteria in detecting the entanglement of non-Gaussian, squeezed number states of two-mode radiation fields has been recently evaluated [166]. Detailed studies of the entanglement properties of non-Gaussian states associated with $SU(1,1)$ active and $SU(2)$ passive optical transformations and the efficiency of the SV NPT criterion to reveal them are currently under way, together with the analysis of their dynamical properties in the presence of noise and decoherence [167].

9.2. Applications, open problems and current perspectives

The centrality of Gaussian states in CV quantum information is motivated not only by their peculiar structural properties which make their description amenable of an analytical analysis, but also by the ability to produce, manipulate and detect such states with remarkable accuracy in realistic, experimental settings.

The scope of this review has been almost entirely theoretical. For reasons of space and time, we cannot discuss in sufficient detail all the proposals and experimental demonstrations concerning on one hand the state engineering of two-, three- and in general N-mode Gaussian states and on the other hand the use of such states as resources for the realization of quantum information protocols. Excellent review papers are already available for what concerns both the optical state engineering of Gaussian and non-Gaussian quantum states of CV systems [168], and the implementations of quantum information and communication with continuous variables [1, 151]. We just mention that, concerning the state engineering, an efficient scheme to produce generic pure N-mode Gaussian states in a standard form not encoding direct correlations between position and momentum operators (and so encompassing all the instances of multimode Gaussian states introduced in the previous sections) has been recently proposed [30]; it enables us to interpret entanglement in this subclass of Gaussian states entirely in terms of the two-point correlations between any pair of modes.

From a practical point of view, Gaussian resources have been widely used to implement paradigmatic protocols of CV quantum information, such as two-party and multiparty teleportation [126, 128, 169–171], and quantum key distribution [138, 139, 172]; they have been proposed for achieving one-way quantum computation with CV generalizations of cluster states [152], and in the multiparty setting they have been proven useful to solve Byzantine agreement [173]. In this respect, one theoretical result of direct interest for the characterization of entanglement in Gaussian states is the qualitative and quantitative *equivalence* [125] between the presence of bipartite (multipartite) entanglement in two-mode (N-mode) fully symmetric Gaussian states shared as resources for a two-party teleportation experiment [126, 169] (N-party teleportation network [128, 170]) and the maximal *fidelity* of the protocol [129, 130], optimized over local single-mode unitary operations performed on the shared resource. In the special case of three-mode, pure GHZ/W states, this optimal fidelity is a monotonically increasing function of the residual contangle, providing the latter with a strong operational interpretation. Based on this equivalence, one can experimentally verify the promiscuous sharing structure of tripartite Gaussian entanglement in such states in terms of the success of two-party and three-party teleportation experiments [143].

Gaussian states are currently considered key resources to realize light-matter interfaced quantum communication networks. It has been experimentally demonstrated how a coherent state of light can be stored onto an atomic memory [174] or teleported to a distant atomic ensemble via a hybrid light-matter two-mode entangled Gaussian resource [175].

Gaussian states play a prominent role in many-body physics, being ground and thermal states of harmonic lattice Hamiltonians [44]. Entanglement entropy scaling in these systems has been shown to follow an area law [176, 177]. In this context, entanglement distribution can be understood by resorting to the 'matrix-product' framework, which results in an insightful characterization of the long-range correlation properties of some harmonic models [140]. Thermodynamical concepts have also been applied to the characterization of Gaussian entanglement: recently, a 'microcanonical' measure over the second moments of pure Gaussian states under an energy constraint has been introduced [178] and employed to investigate the statistical properties of the bipartite entanglement in such states. Under that measure, the distribution of entanglement concentrates around a finite value at the thermodynamical limit and, in general, the typical entanglement of Gaussian states with maximal energy E is *not* close to the maximum allowed by E.

A rather recent field of research concerns the investigation of Gaussian states in a relativistic setting. The entanglement between the modes of a free scalar field from the perspective of observers in relative acceleration has been studied [179, 180]. The loss of entanglement due to the Unruh effect has been reinterpreted in the light of a redistribution of entanglement between accessible and unaccessible causally disconnected modes [180]. Such studies are of relevance in the context of the information loss paradox in black holes [181], in the general framework of relativistic quantum information [182].

From a fundamental point of view, some important conceptual and foundational problems include the operative interpretation and the possible applications of the relative entropy of entanglement for quantum states in infinite-dimensional Hilbert spaces [183], proving whether the Gaussian entanglement of formation coincides with the true entanglement of formation or not, elucidating the hierarchical structure of the distributed Gaussian entanglement between N modes, relative to all possible multipartitions $k \leqslant N$, and extending the quantitative investigation and exploitation of CV entanglement in a relativistic setting to the domain of squeezed and photon-augmented non-Gaussian states [184]. Interest in non-Gaussian entanglement is not restricted to foundational questions. In certain cases non-Gaussian states may prove useful as resources in quantum information protocols, for instance for teleportation with non-Gaussian mixed-state resources closely resembling Gaussian ones [185], quantum information networks with superpositions of odd photon number states [186], and the experimental measurability of some quantifiers of entanglement such as the logarithmic negativity [187]. It has been recently proven that there exist some particular classes of non-Gaussian squeezed states that can be produced and used as entangled resources within the standard Braunstein–Kimble teleportation protocol [126], and allow a sharp enhancement of the fidelity of teleportation, compared to Gaussian resources at fixed degree of squeezing [188].

We have seen how the monogamy constraint establishes a natural ordering and a hierarchy of entanglement of Gaussian states, that goes beyond the frustration effects that arise, for instance, in symmetric graphs [141]. It is then natural to investigate how and in what form monogamy constraints arise in non-Gaussian states of many-body CV systems, and, perhaps even more important, what is the structure of distributed entanglement for hybrid many-body systems composed of qubits, and/or qu*d*its, interacting with single-mode or multimode fields [189]. These investigations may be of special interest for the understanding of decoherence and entanglement degradation in different bath configurations, and the development of possible protection schemes [190]. From this point of view, some interesting hints come from a recent study of the ground-state entanglement in highly connected systems made of harmonic oscillators and spin-1/2 systems [191]. We may expect that this area of research has more surprising, only apparently counterintuitive, results in store, besides the recent finding that, in analogy with finite-dimensional systems, independent oscillators can become entangled when coupled to a common environment [192].

A fundamental achievement would of course be the complete understanding of entanglement sharing in a fully relativistic setting for general interacting systems with a general tensor product structure of individual Hilbert spaces of arbitrary dimension. In a nonrelativistic framework, the investigation of the structure of entanglement in hybrid CV-qubit systems is not only of conceptual importance, but it is relevant for applications as well. Here, we should at least mention a proposal for a quantum optical implementation of hybrid quantum computation, where qubit degrees of freedom for computation are combined with quantum continuous variables for communication [193], and a suggested scheme of hybrid quantum repeaters for long-distance distribution of quantum entanglement based on dispersive interactions between coherent light with large average photon number and single, far-detuned atoms or semiconductor impurities in optical cavities [194]. A hybrid CV memory realized by indirect interactions between different modes, mediated by qubits, has been recently shown to have very appealing features compared to pure-qubit quantum registers [195].

It seems fitting to conclude this review by commenting on the intriguing possibility of observing CV entanglement at the interface between microscopic and macroscopic scales. In this context, it is encouraging that the existence of optomechanical entanglement between a macroscopic movable mirror and a cavity field has been theoretically demonstrated and predicted to be quite robust in realistic experimental situations, up to temperatures well in reach of current cryogenic technologies [196]. These examples together with the few others mentioned above should suffice to convince the reader that continuous-variable entanglement, together with its applications in fundamental quantum mechanics and quantum information, is a very active and lively field of research, where more progress and new exciting developments may be expected in the near future.

Acknowledgments

We would like to acknowledge the continuous exchanges and discussions over the last three years with our friends, colleagues and collaborators at the Quantum Theory Group in Salerno, Lorenzo Albano Farias, Fabio Dell'Anno, Silvio De Siena, Antonio Di Lisi, Salvatore M

Giampaolo, and especially Alessio Serafini, for sharing with us the joy of working together on some important topics covered by the present review. Some of the results that have been reviewed in the present work are the fruit of intense collaborations with Tohya Hiroshima, Marie Ericsson, Ivette Fuentes-Schuller and the group of Claude Fabre at the LKB in Paris. We have greatly benefited from general and specific discussions with Sougato Bose, Sam Braunstein, Hans Briegel, Nicolas Cerf, Ignacio Cirac, Artur Ekert, Jens Eisert, Jaromir Fiuráek, Jeff Kimble, Ole Krueger, Gerd Leuchs, Norbert Lütkenhaus, Klaus Moelmer, Martin Plenio, Eugene Polzik, Anna Sanpera, Peter van Loock, Vlatko Vedral, Frank Verstraete, Ian Walmsley, Reinhard Werner, Michael Wolf and William K Wootters. Stimulating exchanges with Luigi Amico, Fabio Benatti, Ferdinando de Pasquale, Rosario Fazio, Paolo Mataloni, Massimo Palma, Matteo Paris, Saverio Pascazio, Mauro Paternostro, Marco Piani, Francesco Plastina, Lorenza Viola, David Vitali and Paolo Zanardi are gratefully acknowledged. Last but not least, we thank Fabio Benatti and Roberto Floreanini for the happy and collaborative atmosphere created at the Workshop on Quantum Entanglement held in Trieste in June 2006.

References

[1] Braunstein S L and van Loock P 2005 *Rev. Mod. Phys.* **77** 513
[2] Eisert J and Plenio M B 2003 *Int. J. Quantum Inform.* **1** 479
[3] Cerf N, Leuchs G and Polzik E S (ed) 2007 *Quantum Information with Continuous Variables of Atoms and Light* (London: Imperial College Press)
[4] Walls D F and Milburn G J 1995 *Quantum Optics* (Berlin: Springer)
[5] Barnett A M and Radmore P M 1997 *Methods in Theoretical Quantum Optics* (Oxford: Clarendon)
[6] Husimi K 1940 *Proc. Phys. Math. Soc. Japan* **23** 264
[7] Wigner E P 1932 *Phys. Rev.* **40** 749
[8] Glauber R J 1963 *Phys. Rev.* **131** 2766
[9] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[10] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
[11] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[12] Paris M G A, Illuminati J, Serafini A and De Siena S 2003 *Phys. Rev.* A **68** 012314
[13] Simon R 2000 *Phys. Rev. Lett.* **84** 2726
[14] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)
[15] Laurat J, Keller G, Oliveira-Huguenin J A, Fabre C, Coudreau T, Serafini A, Adesso G and Illuminati F 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** S577
[16] Adesso G and Illuminati F 2005 Contribution to Ref. [3] *Preprint* quant-ph/0510052
[17] Simon R, Sudarshan E C G and Mukunda N 1987 *Phys. Rev.* A **36** 3868
[18] Simon R, Mukunda N and Dutta B 1994 *Phys. Rev.* A **49** 1567
[19] Robertson H P 1929 *Phys. Rev.* **34** 163
[20] Schrödinger E 1930 *Berg. Kgl. Akad Wiss.* 296
[21] Serafini A 2006 *Phys. Rev. Lett.* **96** 110402
[22] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[23] Arvind, Dutta B, Mukunda N and Simon R 1995 *Pramana* **45** 471 (*Preprint* quant-ph/9509002)
[24] Wolf M M, Eisert J and Plenio M B 2003 *Phys. Rev. Lett.* **90** 047904
[25] Williamson J 1936 *Am. J. Math.* **58** 141
[26] Serafini A 2004 *PhD Thesis* Università degli Studi di Salerno (www.tampa.phys.ucl.ac.uk/quinfo/people/alessiothesis.pdf)
[27] Adesso G, Serafini A and Illuminati F 2005 *Open Syst. Inform. Dyn.* **12** 189
[28] Serafini A, Illuminati F and De Siena S 2004 *J. Phys. A: Math. Gen.* **37** L21
[29] Huang H and Agarwal G S 1994 *Phys. Rev.* A **49** 52
[30] Adesso G 2006 *Phys. Rev. Lett.* **97** 130502
[31] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)
[32] Bathia R 1997 *Matrix Analysis* (Berlin: Springer)
[33] Bastiaans M J 1984 *J. Opt. Soc. Am.* A **1** 711
[34] Tsallis C 1988 *J. Stat. Phys.* **52** 479
[35] Rényi A 1970 *Probability Theory* (Amsterdam: North-Holland)

[36] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
[37] Adesso G, Serafini A and Illuminati F 2004 *Phys. Rev.* A **70** 022318
[38] Agarwal G S 1971 *Phys. Rev.* A **3** 828
Holevo A S, Sohma M and Hirota O 1999 *Phys. Rev.* A **59** 1820
[39] Holevo A S and Werner R F 2001 *Phys. Rev.* A **63** 032312
[40] Botero A and Reznik B 2003 *Phys. Rev.* A **67** 052311
[41] Giedke G, Eisert J, Cirac J I and Plenio M B 2003 *Quantum Inform. Comput.* **3** 211
[42] Preskill J 1998 *Quantum Information and Computation* (*Lecture Notes for Physics* vol 229) (CA: California Institute of Technology)
[43] Werner R F 1989 *Phys. Rev.* A **40** 4277
[44] Audenaert K, Eisert J, Plenio M B and Werner R F 2002 *Phys. Rev.* A **66** 042327
[45] Serafini A, Adesso G and Illuminati F 2005 *Phys. Rev.* A **71** 023249
[46] Adesso G, Serafini A and Illuminati F 2004 *Phys. Rev. Lett.* **93** 220504
[47] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[48] Heiss D (ed) 2002 *Fundamentals of Quantum Information* (Berlin: Springer)
[49] Audenaert K M R and Plenio M B 2006 *New J. Phys.* **8** 266
[50] Plenio M B and Virmani S 2007 *Quantum Inform. Comput.* **7** 1
[51] Bell J S 1964 *Physics* **1** 195
[52] Clauser J, Horne M, Shimony A and Holt R 1969 *Phys. Rev. Lett.* **23** 880
[53] Gisin N 1991 *Phys. Lett.* A **154** 201
[54] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[55] Yang D, Horodecki M, Horodecki R and Synak-Radtke B 2005 *Phys. Rev. Lett.* **95** 190501
[56] Krüger O and Werner R F (ed) *Open Problems in Quantum Information Theory* (Braunschweig: Institute for Mathematical Physics, Technical University of Braunschweig) (http://www.imaph.tu-bs.de/qi/problems/problems.html)
[57] Terhal B M 2000 *Phys. Lett.* A **271** 319
[58] Brunner N, Gisin N and Scarani V 2005 *New J. Phys.* **7** 88
[59] Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[60] Tapp A, Cleve R and Brassard G 1999 *Phys. Rev. Lett.* **83** 1874
[61] Popescu S and Rohrlich D 1994 *Found. Phys.* **24** 379
[62] Cirel'son B S 1980 *Lett. Math. Phys.* **4** 93
[63] Gisin N 2005 *Preprint* quant-ph/0512168
[64] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 *Phys. Rev.* A **53** 2046
[65] Popescu S and Rohrlich D 1997 *Phys. Rev.* A **56** 3319
[66] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
[67] Vidal G 2000 *J. Mod. Opt.* **47** 355
[68] Bennett C H, Brassard G, Popescu S, Schumacher B, Smolin J A and Wootters W K 1996 *Phys. Rev. Lett.* **76** 722
[69] Gisin N 1996 *Phys. Lett.* A **210** 151
[70] Lewenstein M, Bruss D, Cirac J I, Kraus B, Kuś M, Samsonowicz J, Sanpera A and Tarrach R 2000 *J. Mod. Opt.* **47** 2841
[71] Bruss D 2002 *J. Math. Phys.* **43** 4237
[72] Leuchs G and Beth T (ed) 2003 *Quantum Information Processing* (Berlin: Wiley-VCH)
[73] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[74] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 1
[75] Horodecki P 1997 *Phys. Lett.* A **232** 333
[76] Horodecki M, Horodecki P and Horodecki R 1998 *Phys. Rev. Lett.* **80** 5239
[77] Dür W, Cirac J I, Lewenstein M and Bruss D 2000 *Phys. Rev.* A **61** 62313
[78] DiVincenzo D P, Shor P W, Smolin J A, Terhal B M and Thapliyal A V 2000 *Phys. Rev.* A **61** 62312
[79] Duan L M, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **84** 2722
[80] Werner R F and Wolf M M 2001 *Phys. Rev. Lett.* **86** 3658
[81] Giedke G, Duan L M, Zoller P and Cirac J I 2001 *Quantum Inform. Comput.* **1** 79
[82] Eisert J, Browne D E, Scheel S and Plenio M B 2004 *Ann. Phys.* **311** 431
[83] Eisert J, Scheel S and Plenio M B 2002 *Phys. Rev. Lett.* **89** 137903
[84] Fiurásek J F 2002 *Phys. Rev. Lett.* **89** 137904
[85] Giedke G and Cirac J I 2002 *Phys. Rev.* A **66** 032316
[86] Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 *Phys. Rev. Lett.* **87** 167904
[87] Lewenstein M, Kraus B, Cirac J I and Horodecki P 2000 *Phys. Rev.* A **62** 052310

[88] Gühne O and Lütkenhaus N 2006 *Phys. Rev. Lett.* **96** 170502
[89] Hyllus P and Eisert J 2006 *New J. Phys.* **8** 51
[90] van Loock P and Furusawa A 2003 *Phys. Rev.* A **67** 052315
[91] Horodecki M 2001 *Quantum Inform. Comput.*
Horodecki R, Horodecki P, Horodecki M and Horodecki K 2007 *Preprint* quant-ph/0702225
[92] Christandl M 2006 *PhD Thesis* University of Cambridge
[93] Osborne T J 2007 *Quantum Inform. Comput.* **7** 209 (*Preprint* quant-ph/0402055)
[94] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[95] Terhal B M and Vollbrecht K G H 2000 *Phys. Rev. Lett.* **85** 2625
[96] Vollbrecht K G H and Werner R F 2001 *Phys. Rev.* A **64** 062307
[97] Giedke G, Wolf M M, Krüger O, Werner R F and Cirac J I 2003 *Phys. Rev. Lett.* **91** 107901
[98] Zyczkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 *Phys. Rev.* A **58** 883
[99] Eisert J 2001 *PhD Thesis* University of Potsdam
[100] Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314
[101] Plenio M B 2005 *Phys. Rev. Lett.* **95** 090503
[102] Audenaert K, Plenio M B and Eisert J 2003 *Phys. Rev. Lett.* **90** 027901
[103] Eisert J, Simon C and Plenio M B 2002 *J. Phys. A.: Math. Gen.* **35** 3911
[104] Adesso G, Serafini A and Illuminati F 2004 *Phys. Rev. Lett.* **92** 087901
[105] Adesso G and Illuminati F 2005 *Phys. Rev.* A **72** 032334
[106] Wolf M M, Giedke G, Krüger O, Werner R F and Cirac J I 2004 *Phys. Rev.* A **69** 052320
[107] Rigolin G and Escobar C O 2004 *Phys. Rev.* A **69** 012307
[108] Krüger O 2005 http://www.imaph.tu-bs.de/ftp/krueger/geofnote.html
[109] McHugh D, Ziman M and Buek V 2006 *Phys. Rev.* A **74** 042303
[110] Nielsen M A and Kempe J 2001 *Phys. Rev. Lett.* **86** 5184
[111] Fiuráek J and Cerf N J 2004 *Phys. Rev. Lett.* **93** 063601
[112] Wenger J, sek J F, Tualle-Brouri R, Cerf N J and Grangier P 2004 *Phys. Rev.* A **70** 053812
[113] Ekert A, Alves C M, Oi D K L, Horodecki M, Horodecki P and Kwek L C 2002 *Phys. Rev. Lett.* **88** 217901
[114] Filip R 2002 *Phys. Rev.* A **65** 062320
[115] Oi D K L and {Å}berg J *Phys. Rev. Lett.* **97** 220404
[116] Rigolin G and de Oliveira M C 2006 *Preprint* quant-ph/0608184
[117] D'Ariano G M, Maccone L and Sacchi M F 2005 Contribution to Ref. [3] *Preprint* quant-ph/0507078
[118] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
[119] Eisert J and Plenio M B 1999 *J. Mod. Opt.* **46** 145
[120] Verstraete F, Audenaert K, Dehaene J and De Moor B 2001 *J. Phys. A: Math. Gen.* **34** 10327
[121] Adesso G and Illuminati F 2006 *New J. Phys.* **8** 15
[122] Hiroshima T, Adesso G and Illuminati F 2007 *Phys. Rev. Lett.* **98** 050503
[123] Braunstein S L 1998 *Nature* **394** 47
[124] Verstraete F, Popp M and Cirac J I 2004 *Phys. Rev. Lett.* **92** 027901
[125] Adesso G and Illuminati F 2005 *Phys. Rev. Lett.* **95** 150503
[126] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[127] Bose S, Vedral V and Knight P L 1998 *Phys. Rev.* A **57** 822
[128] van Loock P and Braunstein S L 2000 *Phys. Rev. Lett.* **84** 3482
[129] Braunstein S L, Fuchs C A and Kimble H J 2000 *J. Mod. Opt.* **47** 267
[130] Hammerer K, Wolf M M, Polzik E S and Cirac J I 2005 *Phys. Rev. Lett.* **94** 150503
[131] Adesso G, Serafini A and Illuminati F 2006 *Phys. Rev.* A **73** 032345
[132] Adesso G, Ericsson M and Illuminati F 2006 *Preprint* quant-ph/0609178
[133] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev.* A **61** 052306
[134] Osborne T J and Verstraete F 2006 *Phys. Rev. Lett.* **96** 220503
[135] Terhal B M 2004 *IBM J. Res. Dev.* **48** 71
[136] Adesso G and Illuminati F 2006 *Int. J. Quantum Inform.* **4** 383
[137] Lee S, Chi D P, Oh S D and Kim J 2003 *Phys. Rev.* A **68** 062304
[138] Grosshans F 2005 *Phys. Rev. Lett.* **94** 020504
[139] Navascues M and Acin A 2005 *Phys. Rev. Lett.* **94** 020505
[140] Adesso G and Ericsson M 2006 *Phys. Rev.* A **74** 030305
[141] Wolf M M, Verstraete F and Cirac J I 2004 *Phys. Rev. Lett.* **92** 087903
[142] Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 *Phys. Rev.* A **64** 052303
[143] Adesso G, Serafini A and Illuminati F 2007 *New J. Phys.* at press (*Preprint* quant-ph/0609071)
[144] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314

[145] Araki H and Lieb E H 1970 *Commun. Math. Phys.* **18** 160
[146] Serafini A and Adesso G 2006 *J. Phys. A: Math. Theor.* submitted
[147] Greenberger D M, Horne M A, Shimony A and Zeilinger A 1990 *Am. J. Phys.* **58** 1131
[148] Eisert J and Briegel H J 2001 *Phys. Rev.* A **64** 022306
[149] Ou Y-C 2007 *Preprint* quant-ph/0702127
[150] van Loock P and Braunstein S L 2001 *Phys. Rev.* A **63** 022106
[151] van Loock P 2002 *Fortschr. Phys.* **50** 1177
[152] Menicucci N C, van Loock P, Gu M, Weedbrook C, Ralph T C and Nielsen M A 2006 *Phys. Rev. Lett.* **97** 110501
[153] Wolf M M, Giedke G and Cirac J I 2006 *Phys. Rev. Lett.* **96** 080502
[154] Ourjoumtsev A, Dantan A, Tualle-Brouri R and Grangier P 2007 *Phys. Rev. Lett.* **98** 030502
[155] Kitagawa A, Takeoka M, Sasaki M and Chefles A 2006 *Phys. Rev.* A **73** 042310
[156] Agarwal G S and Biswas A 2005 *New J. Phys.* **7** 211
[157] Nha H and Kim J 2006 *Phys. Rev.* A **74** 012317
[158] Hillery M and Zubairy M S 2006 *Phys. Rev. Lett.* **96** 050503
[159] Shchukin E and Vogel W 2005 *Phys. Rev. Lett.* **95** 230502
[160] Miranowicz A and Piani M 2006 *Phys. Rev. Lett.* **97** 058901
[161] Shchukin E and Vogel W 2006 *Phys. Rev. Lett.* **97** 058902
[162] Miranowicz A, Piani M, Horodecki P and Horodecki R 2006 *Preprint* quant-ph/0605001
[163] Hillery M and Zubairy M S 2006 *Phys. Rev.* A **74** 032333
[164] Shchukin E and Vogel W 2006 *Phys. Rev.* A **74** 030302
[165] McHugh D, Ziman M and Buek V 2006 *Phys. Rev.* A **74** 050306
[166] Dell'Anno F, De Siena S and Illuminati F 2006 *Open Syst. Infor. Dyn.* **13** 383
[167] Albano Farias L, Dell'Anno F, De Siena S and Illuminati F 2007 in preparation
[168] Dell'Anno F, De Siena S and Illuminati F 2006 *Phys. Rep.* **428** 53
[169] Furusawa A, Sørensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
[170] Yonezawa H, Aoki T and Furusawa A 2004 *Nature* **431** 430
[171] Pirandola S and Mancini S 2006 *Laser Phys.* **16** 1418
[172] Grosshans F, Assche G V, Wenger J, Brouri R, Cerf N J and Grangier P 2003 *Nature* **421** 238
[173] Neigovzen R and Sanpera A 2005 *Preprint* quant-ph/0507249
[174] Julsgaard B, Sherson J, Cirac J I, sek J F and Polzik E S 2004 *Nature* **432** 482
[175] Sherson J F, Krauter H, Olsson R K, Julsgaard B, Hammerer K, Cirac J I and Polzik E S 2006 *Nature* **443** 557
[176] Plenio M B, Eisert J, Dreißig J and Cramer M 2005 *Phys. Rev. Lett.* **94** 060503
[177] Cramer M, Eisert J and Plenio M B 2006 *Preprint* quant-ph/0611264
[178] Serafini A, Dahlsten O C O and Plenio M B 2006 *Preprint* quant-ph/0610090
[179] Ahn D and Kim M S 2006 *Preprint* quant-ph/0604007
[180] Adesso G, Fuentes-Schuller I and Ericsson M 2007 *Preprint* quant-ph/0701074
[181] Adesso G and Fuentes-Schuller I 2007 *Preprint* quant-ph/0702001
[182] Peres A and Terno D R 2004 *Rev. Mod. Phys.* **76** 93
[183] Vedral V 2002 *Rev. Mod. Phys.* **74** 197
[184] Illuminati F, Dell'Anno F and Adesso G 2007 in preparation
[185] Lund A P, Ralph T C and van Loock P 2006 *Preprint* quant-ph/0605247
[186] Neergaard-Nielsen J S, Melholt Nielsen B, Hettich C, Moelmer K and Polzik E S 2006 *Phys. Rev. Lett.* **97** 083604
[187] Kitagawa A, Takeoka M, Sasaki M and Chefles A 2006 *Preprint* quant-ph/0612099
[188] Albano Farias L, Dell'Anno F, De Siena S and Illuminati F 2007 in preparation
[189] Tessier T E, Deutsch I H, Delgado A and Fuentes-Guridi I 2003 *Phys. Rev.* A **68** 062316
[190] Illuminati F, Giampaolo S M and Plastina F 2007 in preparation
[191] Ferraro A, Garcia-Saez A and Acin A 2007 *Preprint* quant-ph/0701009
[192] Benatti F and Floreanini R 2006 *J. Phys. A: Math. Gen.* **39** 2689
[193] van Loock P, Munro W J, Nemoto K, Spiller T P, Ladd T D, Braunstein S L and Milburn G J 2007 *Preprint* quant-ph/0701057
[194] Ladd T D, van Loock P, Nemoto K, Munro W J and Yamamoto Y 2006 *New J. Phys.* **8** 164
[195] Paternostro M, Kim M S and Palma G M 2006 *Preprint* quant-ph/0612045
[196] Vitali D, Gigan S, Ferreira A, Böhm H R, Tombesi P, Guerreiro A, Vedral V, Zeilinger A and Aspelmeyer M 2007 *Phys. Rev. Lett.* **98** 030405